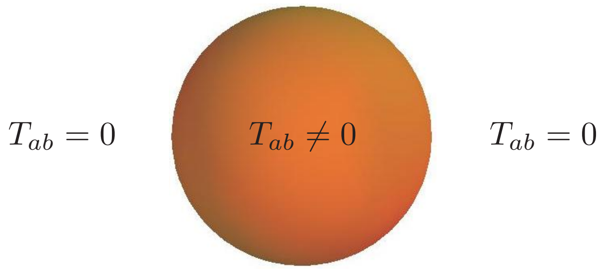
Consider a **static** spherical mass outside which $T_{ab} = 0$, (inside, $T_{ab} \neq 0$).



1

Using symmetry, suggestive coordinates (t, r, θ, ϕ), and *solving Einstein's eq.* the metric is

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\Omega^2$$

where

$$f(r) = 1 - \frac{r_s}{r} \quad \text{and} \quad r_s = \frac{2Gm}{c^2}.$$

2

The constant m has the units of mass, and we interpret it as the mass of the central object that's "creating" this curved geometry exterior to it.

This solution to Einstein's equation was obtained by Karl Schwarzschild almost simultaneously with the final presentation of general relativity. We call it the (exterior) Schwarzschild solution, and we call

3 $r_s = 2Gm/c^2$ the Schwarzschild radius.

In Schwarzschild's (exterior) metric,

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\Omega^2$$

there's a term with $1/f(r)$ in it.

This could be a problem if there's a value of r that makes $f(r) = 0$.

4

(1) Is there? Yes.

$\left(\text{Use } f(r) = 1 - \frac{r_s}{r}.\right)$

If $r = r_s = \frac{2Gm}{c^2}$, then $f(r) = 1 - \frac{r_s}{r_s} = 0$.

(2) There's a second problem, this time with $f(r)$ itself. At what r?

$r = 0$.

5

The values $r = r_s$, and $r = 0$ are singularities of the Schw. metric and need to be understood.

Yet, for decades people dismissed these problems.

Schwarzschild's metric describes geometry *outside* a spherical object. For the problems to be real, an object would have to have a radius $r \leqslant 2Gm/c^2$ in the first case, and $r = 0$ in the second.

6

ADDITIONAL NOTES

For the earth,

$$r_s = \frac{2Gm}{c^2} = \frac{2(6.7 \times 10^{-11})(6.7 \times 10^{24})}{(3 \times 10^8)^2} = \frac{2(6.7)(6.7)}{3^2} \frac{10^{13}}{10^{16}} \approx 10^{-2}\text{m} = 1\text{cm}.$$

That's absurdly small. What about for other heavenly objects, such as the sun – or you?

7

$G = 6.7 \cdot 10^{-11}$ (SI units), and $c = 3 \times 10^8$m/s. Complete the following table to get $r_s = 2Gm/c^2$ (in km) to compare it with R (actual radius):

Object	m (kg)	r_s (km)	R (km)
Sun	$2 \cdot 10^{30}$	3.0	700,000
W. dwarf	$3 \cdot 10^{30}$	4.5	$\sim$ 7,000
You	50	$7 \cdot 10^{-29}$	$\sim$ 0.0005

8 Your r_s is about 7 hundred-billionth the radius of a proton.

The crazily small size of the Schwarzschild radius (coupled with the apparent singularity at $r = r_s$) led to the belief that every physical object must have a radius greater than its Schwarzschild radius.

Einstein himself came to believe this after a study he made of the Schwarzschild solution.

9

ON A STATIONARY SYSTEM WITH SPHERICAL SYMMETRY CONSISTING OF MANY GRAVITATING MASSES

BY ALBERT EINSTEIN

(Received May 10, 1939)

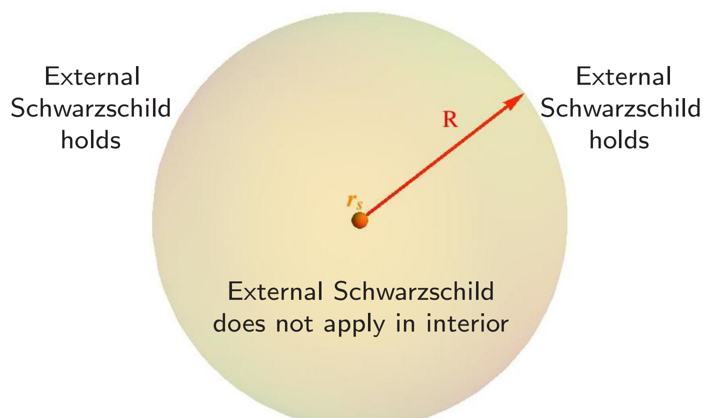
After studying a few scenarios (particles, continuous distributions of matter,. . .) Einstein concludes:

10

The essential result of this investigation is a clear understanding as to why the "Schwarzschild singularities" do not exist in physical reality. Although the theory given here treats only clusters whose particles move along circular paths it does not seem to be subject to reasonable doubt that more general cases will have analogous results. The "Schwarzschild singularity" does not appear for the reason that matter cannot be concentrated arbitrarily. And this is due to the fact that otherwise the constituting particles would reach the velocity of light.

This investigation arose out of discussions the author conducted with Professor H. P. Robertson and with Drs. V. Bargmann and P. Bergmann on the mathematical and physical significance of the Schwarzschild singularity. The problem quite naturally leads to the question, answered by this paper in the negative, as to whether physical models are capable of exhibiting such a singularity.

11



12

ADDITIONAL NOTES

What might an "interior metric" look like? With the assumption that the density is constant (as in a solid object), the indefatigable Schwarzschild arrived at that too. Let R be the radius of the object, and m its mass. Remember that

$$f(r) = 1 - \frac{r_s}{r} \quad \text{and} \quad r_s = \frac{2Gm}{c^2}.$$

Let

$$g(r) = 1 - \frac{r^2 r_s}{R^3}$$

13

(3) What's $f(R)$? $1 - r_s/R$

(4) What's $g(R)$? $1 - r_s/R$

(5) Do they match at $r = R$? Yes!!!

The interior Schwarzschild metric is

$$ds^2 = \frac{1}{4}\left(3\sqrt{f(R)} - \sqrt{g(r)}\right)^2 dt^2 - \frac{1}{g(r)} dr^2 - r^2 d\Omega^2$$

14

(6) Show that the exterior and interior metrics match at $r = R$.

We only need to look at the dt term:

$$\frac{1}{4}\left(3\sqrt{f(R)} - \sqrt{g(R)}\right)^2 = \frac{1}{4}\left(3\sqrt{f(R)} - \sqrt{f(R)}\right)^2 = \frac{1}{4}\left(2\sqrt{f(R)}\right)^2 = f(R)$$

15

What about singularities at $r = r_s$, and $r = 0$?

(7) What's $g(r_s)$? $1 - r_s^3/R^3$

(8) If $R > r_s$, is this zero? No darn way!!!

So, no metric component is singular at $r = r_s$.

(9) What's $g(0)$? 1

So, no metric component is singular at $r = 0$.

There are no interior singularities. Problem solved.

Yeah. You wish.

16

All of this would be fine if there were no physical possibility that a physical object can have $r \leqslant r_s$.

Mathematics and physics co-exist with contradictions all the time.

The idea of a "point particle" is both impossible (infinite density?) and useful.

Chandrasekhar screwed up the situation.

17

Einstein was not alone in his disinclination to believe that such highly concentrated forms of matter could exist. Although gravity is universally attractive, most objects have forces that act as internal pressures to prevent gravitational collapse.

(10) Your feet are attracted to your head (gravitationally). What prevents you undergoing gravitational collapse? Electromagnetic forces.

18

ADDITIONAL NOTES

(11) What prevents a star from undergoing gravitational collapse?
Outward pressure from nuclear interactions.

This means that when a star exhausts its nuclear fuel something else must take over, or gravitation wins.

19

In the 1920s a class of stars at the end of stellar life, called white dwarfs, became known.

"Electron degeneracy pressure" was thought to be the force that supported white dwarfs.

A 19-year-old student from India, Subramaniam Chandrasekhar, was studying white dwarf structure on the way to England. In those days the trip was made by boat, so he had plenty of time.

20

Chandrasekhar found that only white dwarfs whose masses were less than about 1.4 solar masses were stable. The rest would collapse under their own gravity.

This implied that there was nothing to stop them from becoming smaller than their Schwarzschild radii.

21

The mass of a typical white dwarf is $\sim 3\times10^{30}$ kg, $c \approx 3\times10^{8}$ m/sec, & $G \approx 6.7\times10^{-11}$ m^3/kg·sec^2. In these units, as we've seen, r_s for the white dwarf (nearest km) is:

$$r_s \approx \frac{2(6.7\times10^{-11})(3\times10^{30})}{9\times10^{16}} = \frac{2(6.7)(3)}{9}10^{3} \approx 4.5 \text{ km}.$$

22

The radius of such a typical white dwarf is originally about $7{,}000$ km, about that of earth.

Although compressing it to 4.5 km is not as far-fetched as compressing Earth to a teaspoon, it's still difficult to grasp.

23

The great British astronomer, Arthur Stanley Eddington, was one of those who experienced this difficulty.

Although Chandrasekhar published his result in 1931, Eddington attacked the result repeatedly in lectures and in print, calling it absurd.

24

ADDITIONAL NOTES

He said

"The star apparently has to go on . . . contracting and contracting until, I suppose, it gets down to a few kilometers radius . . . there should be a law of Nature to prevent the star from behaving in this absurd way."

25

From "Home is Where the Wind Blows," Fred Hoyle, University Science Books, 1994 (p. 123-24):

". . . a great deal of blood was spilt in 1937 on a weekly basis. . . no less a person than Eddington had spilt a vatful . . . on the occasion in question, he mistakenly tried to claim that black was white with truly awful results. . . there was no permanent condition for a large enough quantity of nonrotating matter, a quantity now known as the Chandrasekhar limit,. . . Eddington thought that [this] should be a physical impossibility, so he inferred that the correct formula should be something else."

26

From the AIP Oral history archive (http://www.aip.org/history/ohilist/4551_1.html):

Chandrasekhar: "at the end of the meeting, everybody came and said to me, 'Too bad. Too bad.' The other astronomers were certain that my work was wrong because Eddington had said so."

Interviewer: 'Too bad' meaning, 'Too bad that you had got it wrong'?

Chandrasekhar: "Yes."

27

Chandrasekhar: "In many ways, thinking back over those times, I am sort of astonished that I was never completely crushed by these Stalwarts. You know, none of these people would accept my work, astronomers wouldn't accept it and finally in 1938, I decided that there was no good my fighting all the time, that I am right and that the others were all wrong. I would write a book. I would state my views. And I would leave the subject. That's exactly what I did."

28

Chandrasekhar: "It is hard for people to realize what an incredibly dominating position Eddington had during his life. For example, Shapley told me this: in 1936, they had a tricentennial at Harvard, and, Shapley said, they sent a circular around to American astronomers, to rank astronomers so they could give honorary degrees. And he said that Eddington was the first in every single list he received! ... the fact is that there was not a single astronomer in the thirties who would not with unanimity have said that Eddington is the greatest living astronomer. He had an absolutely dominating position."

29

The objects that collapse without limit that Chandrasekhar (almost) predicted are what we now call black holes.

They began to be widely accepted only in the 1960s when pulsars were discovered and were confirmed to be neutron stars. Such objects have masses between one and two solar masses (so r_s between 3 and $6\,$km) and radii of about $12\,$km.

30

ADDITIONAL NOTES

Part of the problem that people had at the time in accepting the existence of objects smaller than their Schwarzschild radii is the they were confused about what happens at r_s.

The problem was an over reliance on coordinates.

31

We've classified vectors into three types:

$$\text{If } g_{ab}V^aV^b \text{ is } \begin{cases} > 0\,, & V^a \text{ is timelike} \\ = 0\,, & V^a \text{ is null} \\ < 0\,, & V^a \text{ is spacelike} \end{cases}$$

32

(12) What are the components of a vector, V^a, pointing in the positive t direction in Schwarzschild?

$V^a = (V^0, 0, 0, 0)$, where we can take $V^0 = 1$.

(13) If t is supposedly the time coordinate, does your instinct say that V^a is timelike, spacelike or null? Timelike.

33

(14) Find $g_{ab}V^aV^b$ (g_{ab} is Schwarzschild metric).

$$\begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & -\frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\,sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= f(r)$$

34

(15) Is $f(r) > 0$ for all values of r? No.

(16) If not, for what values?

$$f(r) = 1 - \frac{2M}{r} \begin{cases} > 0 & \text{for} \quad r > 2M \\ = 0 & \text{for} \quad r = 2M \\ < 0 & \text{for} \quad r < 2M \end{cases}$$

35

(17) What do you conclude about V^a, and therefore the "time" coordinate t?

V^a is timelike for $r > 2M$, null for $r = 2M$, and spacelike for $r < 2M$.

This means t is a time coordinate for $r > 2M$, but *a space coordinate for $r < 2M$*.

36

ADDITIONAL NOTES

(18) What does this teach us?

You cannot *start* by assuming which coordinate is time.

(19) If time does not point in the t direction for $r < 2M$, in what direction does it point?

In the r direction! Easy to check that $(0, \pm 1, 0, 0)$ are both timelike for $r < 2M$.

37

This confusion about time occurs only for $r \leqslant 2M$.

Since the Schwarzschild geometry is the *exterior* geometry of a spherically symmetric mass, there would be no issue if objects are always larger than their Schwarzschild radii.

38

Dependence on Coordinates

"To those of us educated in the field in the last twenty years or so, the fumblings and misunderstandings of the years from 1915 to 1950 seem remote and strange. The emphasis on asking questions about relations between physical observables, rather than coordinate relations, that we have received in our training, makes the long disputes over questions like the reality of the gravitational wave, or the physical reality of the $r = 2M$ Schwarzschild singularity hard to understand."

– Bill Unruh in 1979 in "Some Strangeness in the Proportion," An Einstein Centennial Symposium.

39

"Up to about 1960, most work in general relativity was performed in terms of metric components in a single coordinate system. This . . . led to confusion and misunderstanding . . . A particular example of this was the Schwarzschild solution . . . The metric components, in the normally used coordinate system, become singular at the 'Schwarzschild radius' $r = 2M$ . . . its fictitious character was not generally recognized until 1959. . .

– Stephen Hawking at the same symposium.

40

What does Hawking mean by "fictitious character"?

It's a research-level task. People who find "better" coordinates publish them. Let's start with $r = r_s$.

We follow convention and introduce $M = Gm/c^2$. Then $r_s = 2M$, and

$$f(r) = 1 - \frac{r_s}{r} = 1 - \frac{2M}{r}.$$

41

Consider the coordinate transformation

$$v = t + r + 2M\, ln(r - 2M).$$

(20) What is dv?

$$dv = dt + dr + \frac{2M}{(r-2M)} dr$$

42

ADDITIONAL NOTES

(21) Group the dr terms and simplify till you can express your answer using $f(r)$ from the standard Schwarzschild solution (or drop from exhaustion).

$$dv = dt + \left[1 + \frac{2M}{(r-2M)}\right] dr$$
$$= dt + \left[\frac{r - 2M + 2M}{r - 2M}\right] dr$$

43

$$= dt + \left[\frac{r}{r-2M}\right] dr$$
$$= dt + \left[\frac{r/r}{(r-2M)/r}\right] dr$$
$$= dt + \left[\frac{1}{1-2M/r}\right] dr$$
$$= dt + \frac{1}{f(r)} dr.$$

44

(22) Solve for dt, then find dt^2.

$$dt = dv - \frac{1}{f(r)} dr$$

$$dt^2 = dv^2 - \frac{2}{f(r)} dv\, dr + \frac{1}{\left(f(r)\right)^2} dr^2$$

45

(23) Plug this into the r-t part of the Schwarzschild metric

$$ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2$$

and say bye-bye to the coordinate t.

46

$$ds^2 = f(r)\left[dv^2 - \frac{2}{f(r)} dv\, dr + \frac{1}{\left(f(r)\right)^2} dr^2\right] - \frac{1}{f(r)} dr^2$$
$$= f(r) dv^2 - 2 dv\, dr + \frac{1}{f(r)} dr^2 - \frac{1}{f(r)} dr^2$$
$$= f(r) dv^2 - 2 dv\, dr$$

47

Remember that $f(r) = 1 - 2M/r$.

In these new v-r coordinates, does the metric still have a singularity (infinity) at

(24) $r = 2M$? No.

(25) $r = 0$? Yes.

48

ADDITIONAL NOTES

The singularity at $r = 2M$ is "fictitious." It can be removed by a coordinate transformation.

These coordinates were known early in relativity (introduced, oddly, by Eddington, 1924).

That they removed the $r = 2M$ singularity was not understood till later (Finkelstein, 1958).

49

"One reason the apparent singularity at $r{=}2M$ was not investigated more throughly was that it was generally thought to be unphysical: no 'real' body would ever become so compressed that it would be inside its Schwarzschild radius. . . "

– Stephen Hawking in 1979 in "Some Strangeness in the Proportion," An Einstein Centennial Symposium.

What about the singularity at $r = 0$?

50

The $r = 0$ singularity is real, and can't be removed by a coordinate transformation. How do we know?

There are numbers that can be computed from g_{ab} that have the same value in any coordinate system. They are called scalar invariants. The curvature R is one such invariant. It may differ from point to point, but at any given point it has the same value in all coordinates.

51

If we examine the behavior of certain curvature invariants (not R, in this case, but others) near $r = 2M$ we get a clue that the apparent problem there is not real, but that the problem at $r = 0$ is serious. There are invariants that are well-behaved at $r = 2M$, but approach ∞ as $r \to 0$.

52

The real singularity at $r = 0$ occurs in this particular Schwarzschild model, where we have assumed exact spherical symmetry. Nothing in the real world is exactly spherical.

What happens without symmetry?

53

The

$$(v, r, \theta, \phi)$$

coordinates are called Eddington-Finkelstein coordinates, after Eddington who introduced them and Finkelstein who saw their significance.

(There are two sets, ingoing and outgoing.)

The singularity at $r = 2M$ is removed in these coordinates, but the one at $r = 0$ remains.

54

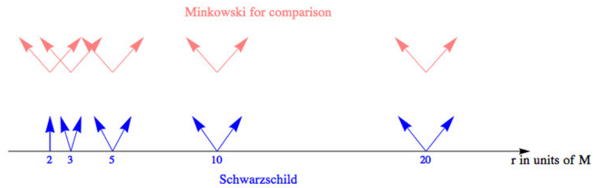
ADDITIONAL NOTES

The full (4-d) Eddington-Finkelstein metric in matrix form.

$$g_{ab} = \begin{pmatrix} f(r) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\, sin^2\,\theta \end{pmatrix}$$

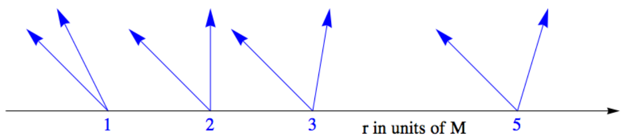
55

In (t, r) coordinates you cannot see what is happening at $r = 2M$:



(v, r) coordinates give a better picture of how radial light rays behave near, at, and inside $r = 2M$.

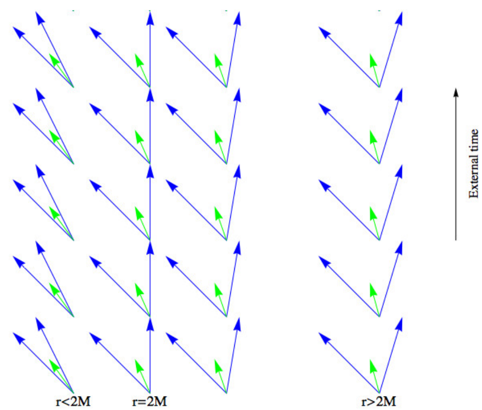
56



We see that there is a well behaved "light cone" at $r = 2M$ in these coordinates.

This is a snapshot at one time. We can "stack" such pictures to get a spacetime view.

57



58

This picture illustrates several important features of the Schwarzschild solution.

1) The long (blue) arrows represent null vectors. They give the light cones at each point and are potential directions for the trajectories of massless particles (or for light rays).

59

2) The short (green) arrows "inside" represent timelike vectors. They are potential directions for the trajectories of material ("massive") particles.

3) "External time" agrees with the direction of the short arrows for $r \gg 2M$ but not for $r \leqslant 2M$.

("External time:" for large r, Schwarzschild spacetime approaches Minkowski, so time here agrees with Minkowski.)

60

ADDITIONAL NOTES

4) The surface at $r = 2M$ is a "one way membrane", built out of pure geometry. It's the surface of what we now call a black hole. You can go in but you can't come out.

It is called the event horizon. Inside it, at $r = 0$, there is a real singularity.

All this understanding emerged only in the 1960s.

61

(26) So, is increasing r the future or is decreasing r the future?

Decreasing.

(27) Where does the future lie at $r = 2M$?

Into the black hole.

62

Blackholes: Friends or Frenemies?

A black hole sounds bad news. Should we run?

Well outside a black hole you have none of the getting-sucked-in-and-never-being-seen-again-by-your-friends black hole headaches.

If the sun broke bad and suddenly decided to go blackhole, we'd experience no gravitational setbacks.

63

(28) I said "getting-sucked-in-and-never-being-seen-again-by-your-friends," not "getting-sucked-in-and-never-seeing-your-friends-again." Why?

Because you can check IN (and friends can send you chocolate), not check OUT.

64

What can you do where?

$r > 6M$: can safely orbit a black hole without any rocket propulsion (geodesic motion or "free fall").

$3M < r \leqslant 6M$: there are circular free-fall orbits, but they're unstable.

$2M < r \leqslant 3M$: there are no circular free-fall orbits, but you can accelerate in and out. There are closed orbits of light at $r = 3M$.

$r \leqslant 2M$: you're most likely dead.

65

Detection of Black Holes

- ▷ As the invisible partner in a gravitational dance (such as a binary).
- ▷ Accretion disks.
- ▷ Radiation from infalling matter.
- ▷ Jets.

66

ADDITIONAL NOTES

The Densities of Black Holes

The radius of a black hole is $r_s = 2Gm/c^2$.

The volume of a sphere is $\frac{4\pi}{3}r^3$.

For a black hole the volume is $\frac{4\pi}{3}\left(\frac{2Gm}{c^2}\right)^3$.

The density is $\frac{m}{\frac{4\pi}{3}\left(\frac{2Gm}{c^2}\right)^3} = \frac{3c^6}{32\pi G^3}\frac{1}{m^2}$.

67

(29) The mass of the sun is $\sim 2 \times 10^{33}$ gms and its $r_s \approx 3 \times 10^5$ cm. What would its density be were it to shrink to a black hole?

$$\frac{2 \times 10^{33}}{(4/3)\pi(3 \times 10^5)^3} \approx 2 \times 10^{16} \text{g/cm}^3.$$

(20 million billion times the density of water.)

68

An object that's two hundred million times as massive as the sun will have a density that smaller by a factor of

$$\frac{1}{4 \times 10^{16}}$$

(30) Why? Down by $1/m^2$.

(31) What is the density?

$$\frac{2 \times 10^{16}}{4 \times 10^{16}} = \frac{1}{2} \text{ g/cm}^3$$

That's half the density of water.

69

Are there such supermassive objects?

Yes. In black holes at centers of galaxies.

Here are some (in solar masses, $M_\odot$):

Milky Way: $\sim$ four million $M_\odot$

Andromeda: $\sim$ one hundred million $M_\odot$

M87: $\sim$ six billion $M_\odot$

We know from the dance of nearby stars.

70

ADDITIONAL NOTES