

# Arvind Borde / PHY 19, Week 3: Momentum and Energy

## §3.1 Inertial Frames

Class of frames which

a) Move uniformly wrt each other, and in which

b) Newton's First Law holds.

("No force, no acceleration.")

We assume our frames are inertial, but we'll see

1 there are problems with the concept.

In this frame, momentum is conserved:

Initial:  $\vec{p} = m\vec{v} + m(-\vec{v}) = 0.$

Final:  $\vec{p} = 0.$

Now, consider a frame moving to the right with velocity  $\vec{v}$ .

(1) What is the initial momentum of the left ball in this frame? \_\_\_\_\_

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(4) What is the final velocity of the stuck-ball pair in the new frame? \_\_\_\_\_

(5) What's the final momentum here? \_\_\_\_\_

(6) Is momentum conserved in this frame? \_\_\_\_\_

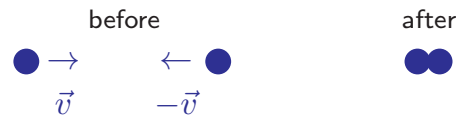
In order to preserve the conservation of momentum, we need a new definition of momentum.

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## §3.2 Momentum

We need to redefine momentum as  $\vec{p} = \gamma(v)m\vec{v}$  because otherwise \_\_\_\_\_.

Example: Two balls of equal mass that collide and stick (inelastic collision) in a frame in which they have equal and opposite initial velocities:



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With respect to the new frame, the old frame is moving in the negative  $x$  direction with speed  $v$ .

(2) Using relativistic addition of velocities, what is the initial velocity of the right ball in this frame?

(3) What is the total initial momentum?

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## §3.3 Newton's Second law

This law is expressed in introductory physics as \_\_\_\_\_.

If  $m$  is assumed constant, and  $\vec{p} = m\vec{v}$ , we can re-express this as

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ADDITIONAL NOTES

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In relativity, although these *steps* are invalid, the final result still holds:  $\vec{F} = d\vec{p}/dt$ .

One reason that this is the “correct” law is that it enforces the conservation of momentum.

If the total external force on a system is zero, then

$$\frac{d\vec{p}}{dt} = \frac{d(\gamma(v)m\vec{v})}{dt} = 0$$

7 in other words,  $\vec{p}$  is constant (aka “conserved”).

(7) Calculate  $d(\gamma(v))/dt$  in terms of  $dv/dt$ .

$$\frac{d(\gamma(v))}{dt} =$$

(8) Confining ourselves to magnitudes  $p = \gamma mv$ .

Calculate  $dp/dt$ .

$$\frac{dp}{dt} = \frac{d(\gamma mv)}{dt} =$$

Therefore,

### §3.4 Energy

If a force moves an object from  $x_1$  to  $x_2$  on the  $x$  axis, it does work given by

$$\begin{aligned} W &= \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} \frac{dp}{dt} dx \\ &= \int_{x_1}^{x_2} \gamma^3 m \frac{dv}{dt} dx \end{aligned}$$

11 from the previous formula for  $dp/dt$ .

But,

$$\frac{dv}{dt} dx = dv \frac{dx}{dt} = v dv$$

Assuming that the particle being moved by the force is at rest at  $x_1$  and has velocity  $v$  at  $x_2$ ,

$$\begin{aligned} W &= \int_{x_1}^{x_2} \gamma^3 m \frac{dv}{dt} dx = m \int_0^v \gamma^3 v dv \\ &= m \int_0^v \frac{v dv}{(1 - v^2/c^2)^{(3/2)}} \end{aligned}$$

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#### ADDITIONAL NOTES

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Using the substitution  $w = \frac{m_0 c^2 (\gamma - 1)}{v}$ ,  
 we have  $dw = \frac{m_0 c^2 \gamma^3 v}{c^2} dv$ . So, we get

$$W =$$

Since the work done by a force in setting a particle of rest mass  $m_0$  in motion (from zero initial velocity) equals the KE of the particle, we get

$$E_{\text{kin}} = m_0 c^2 (\gamma - 1)$$

which may also be rewritten as

$$m_0 c^2 \gamma = E_{\text{kin}} + m_0 c^2$$

where we think of  $\gamma m_0 c^2$  as the total energy, and  $m_0 c^2$  as the “mass-energy.”

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(9) In units where  $c = 1$  write the speed  $v$  as a function of  $E_{\text{kin}}/m$  in both the non-relativistic ( $E_{\text{kin}} = 1/2 m v^2$ ) and relativistic cases.

Non-rel:

Rel:

In both cases (Relativistic and Non-)

(10) what is  $v$  at  $E_{\text{kin}} = 0$ ?

(11) how does  $v$  behave as  $E_{\text{kin}} \rightarrow \infty$ ?

Non-rel:

Rel:

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(12) An electron has a (rest) mass of  $9.11 \times 10^{-31}$  kg. what is its “rest energy” ( $mc^2$ )?

Now, an electron Volt (eV) is the energy obtained by an electron when it is accelerated by a potential difference of 1 V.

$$1\text{eV} = 1.60 \times 10^{-19}\text{J}$$

(13) What is the rest energy of an electron in eV?

This is a standard way now to express the masses of elementary particles, as eV, MeV, GeV, etc.

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ADDITIONAL NOTES

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§3.5 Momentum and Energy

We can rewrite the expression for energy using momentum rather than velocity, using  $E_{\text{tot}} = \gamma mc^2$  and  $p = \gamma mv$ .

(14) What is  $E_{\text{tot}}^2 - p^2 c^2$ ?  
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So we have

$$E_{\text{tot}}^2 = p^2 c^2 + (mc^2)^2$$

where  $m$  is the “rest mass.”

This expression applies even to particles *with zero rest mass*, such as photons. For these we get

$$E = pc.$$

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Also, since the rest mass,  $m$ , and the speed of light,  $c$ , are the same in all reference frames,

$$E^2 - p^2 c^2$$

is an invariant under Lorentz transformations.

Remember the invariance of the \_\_\_\_\_.

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§3.6 Conservation of Mass-Energy

The separate conservation laws of mass and energy in pre-relativistic physics are combined onto a single, unified law of conservation of mass-energy (i.e.,  $E_{\text{tot}} = \gamma mc^2$ ) in special relativity:

“The sum of the mass-energy of a system of particles before interaction must equal the sum of the mass-energy of the system after interaction.”

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Consider the sticky collision discussed earlier. Conservation of mass energy, in the first frame we looked at, says

$$\gamma(v)mc^2 + \gamma(-v)mc^2 = \gamma(0)Mc^2$$

where we are *not* assuming that  $M = m + m$ .

Since  $\gamma(v) = \gamma(-v)$  and  $\gamma(0) = 1$ ,

$$M = 2m\gamma(v).$$

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(15) Is  $M = 2m$  or what (and why)?  
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(16) What is  $(M - 2m)c^2$ ?

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ADDITIONAL NOTES

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**§3.7 Conservation of Relativistic Momentum**

Consider the inelastic collision we looked at earlier.

In the first frame

Initial:  $p = \gamma m v + \gamma m (-v) = 0$ .

Final:  $p = \gamma M \times 0 = 0$ .

Relativistic momentum is conserved.

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(17) What is  $1 - v_R^2/c^2$ ?

(18) What is  $\gamma(v_R)$ ?

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In the second frame:

Initial momentum of left ball:  $\gamma m \times 0 = 0$ .

Initial momentum of right ball:  $\gamma(v_R) m v_R$

Total initial momentum:  $\gamma(v_R) m v_R$ , where

$$v_R = \frac{-2v}{1 + v^2/c^2}$$

Final momentum:  $\gamma(-v) M \times (-v) = -\gamma(v) M v$ .

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(19) What is the total initial momentum?

(20) What is the total final momentum?

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ADDITIONAL NOTES

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