§2.1 Summary of Part I of 1905 Paper

Einstein's 1905 paper had two parts:

I. Kinematical ____________________

II. Electrodynamical

It was the kinematical part that contained the basic new ideas of what we now call the __________ ______________

1 It had 5 sections:

§1. Definition of Simultaneity

§2. On the Relativity of Lengths and Times

____

§3. Theory of the Transformation of Co-ordinates...

____

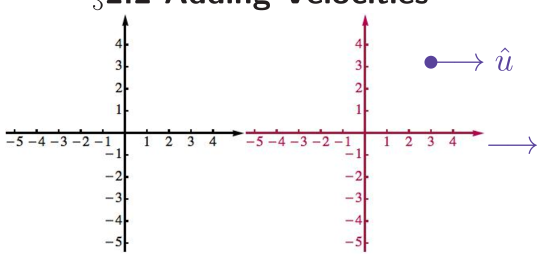
§4. Physical Meaning of the Equations Obtained...

____

§5. The Composition of Velocities

2 ______________

§2.2 Adding Velocities



Hatted frame initially coincides with unhatted, and is moving with speed v relative to it; a blob is moving with speed $\hat{u}$, *as measured in the hatted frame*, starting from origin at

3 $\hat{t} = 0 = t$.

What is the speed, u, of the blob as measured in the unhatted frame?

Galilean commonsense says...

(1) What? ______________.

4

We turn to our new bff's, the Lorentz transformations:

$$\hat{t} = \gamma(t - vx/c^2)$$

$$\hat{x} = \gamma(x - vt)$$

A bit of algebra (or a symmetry argument) gives the reverse Lorentz transformations:

$$t = \gamma(\hat{t} + v\hat{x}/c^2)$$

5
$$x = \gamma(\hat{x} + v\hat{t})$$

(2) What is the $\hat{x}$ coordinate of the blob?

$\hat{x} =$ __________

(3) What do you get when you plug this into the reverse Lorentz transformations?

$t =$ ________________________

$x =$ ____________________

6

ADDITIONAL NOTES

(4) What's u (speed of blob in unhatted frame) in unhatted coordinates? ______________

(5) Using the results of Q3, what does this ratio work out to?

$$u = \frac{x}{t} =$$

7

§ 5. The Composition of Velocities

In the system k moving along the axis of X of the system K with velocity v, let a point move in accordance with the equations

y, z, t and the corresponding quantities of k', which differ from the equations found in § 3 only in that the place of "v" is taken by the quantity

$$\frac{v+w}{1+vw/c^2};$$

from which we see that such parallel transformations—necessarily—form a group.

We have now deduced the requisite laws of the theory of kinematics corresponding to our two principles, and we proceed to show their application to electrodynamics.

8

(6) If the hatted frame and the blob are both moving at speeds much smaller than that of light ($v \ll c$, $\hat{u} \ll c$), show that the relativistic addition of velocities reduces to our discarded commonsense.

9

If the blob were a blob of light,

(7) What would $\hat{u}$ be? ____________

(8) What would u be (from the formula)?

$$u =$$

10

§2.3 Momentum, Mass and Energy

It follows that our concepts of momentum and energy have to change if we want the "laws of mechanics" to be the same for all observers moving uniformly with respect to each other.

What are these laws that we wish to save?

The most important are __________________.

11

(9) What's a conservation law?

____

____

(10) What are examples of conservation laws?

- ____________________________

____

- ________________________

____

12

ADDITIONAL NOTES

§2.3.1 Momentum

In order to preserve the conservation of momentum, we need to redefine it in relativity.

(11) What is the usual (Newtonian) definition of momentum for an object whose mass is m_0 and velocity is $\vec{v}$? ________________

13

If momentum has to be conserved, the relativistic law of addition of velocities dictates that we redefine it as

Does it make more sense to view the γ factor as "belonging" to the velocity or to the mass?

It's possible to make a case either way.

14

§2.3.2 Mass

If you view γ as attached to the mass, then an object whose mass at rest (________________) is m_0 will have an effective mass when moving at speed v of

____

15

(12) How does $m(v)$ behave as $v \to c$?

____

____

____

(13) At low speeds what does the relativistic momentum formula, $\vec{p} = \gamma m_0 \vec{v}$, reduce to?

____

16 ____

§2.3.3 Energy

(14) What's the Newtonian formula for the kinetic energy of an object (mass m_0, speed v)?

$-$

____

(15) What (do you think) is the relativistic formula for KE?

17 _________________________

What does the relativistic kinetic energy formula reduce to at low speeds?

We can't simply use $\gamma \approx 1$. Rewrite the definition:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = (1 - v^2/c^2)^{\_\_\_\_}$$

This can be expanded via the binomial theorem.

18

ADDITIONAL NOTES

Binomial Theorem

$$(x+y)^r = x^r + rx^{r-1}y^1 + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \ldots$$

where

$$k! = k(k-1)\cdots 1 \quad \text{with} \quad 0! \equiv 1.$$

19

(16) What are these

$(x+y)^1 =$

$(x+y)^2 =$

20

(17) What are the first two terms in

$$\gamma = (1 - v^2/c^2)^{-1/2}?$$

21

(18) Plugging these two terms into the relativistic kinetic energy formula, what do you get?

22

The equation

$$E_{\text{kin}} = m_0 c^2(\gamma - 1)$$

may be rewritten as

$$\gamma m_0 c^2 = \underline{\underline{\hspace{6cm}}}$$

Defining the left-hand side of the equation as the total energy, E, for a particle at rest we get . . .

23

DOES THE INERTIA OF A BODY DEPEND UPON ITS ENERGY-CONTENT?

BY A. EINSTEIN

September 27, 1905

The results of the previous investigation lead to a very interesting conclusion, which is here to be deduced.

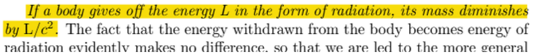
I based that investigation on the Maxwell-Hertz equations for empty space, together with the Maxwellian expression for the electromagnetic energy of space, and in addition the principle that:—

The laws by which the states of physical systems alter are independent of the alternative, to which of two systems of coordinates, in uniform motion of parallel translation relatively to each other, these alterations of state are referred (principle of relativity).

24

ADDITIONAL NOTES

It was a short paper (three pages) at the end of which Einstein concluded


If a body gives off the energy L in the form of radiation, its mass diminishes by L/c². The fact that the energy withdrawn from the body becomes energy of radiation evidently makes no difference, so that we are led to the more general

In other words, $m = L/c^2$, where L is the energy.

Or, as we know it, $E = mc^2$.

25

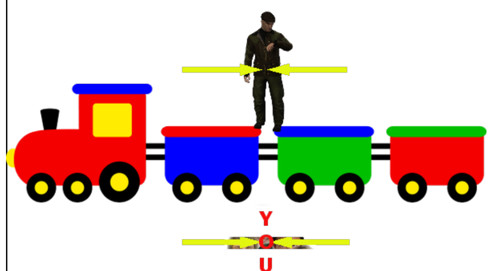
§2.4 "Paradoxes"

(19) *Einstein's Mirror*: If you hold a mirror at arms length and run backward (still holding it) at close to the speed of light, will it take longer for you to see yourself compared to when you were at rest?

26

(20) *Einstein's Train*: A train passes by you with a guard in the middle. Just as the guard passes you, flashes of light emitted from the front and the back of the train reach both you and the guard. Both of you agree on this. Do you agree on when the flashes were emitted?

27

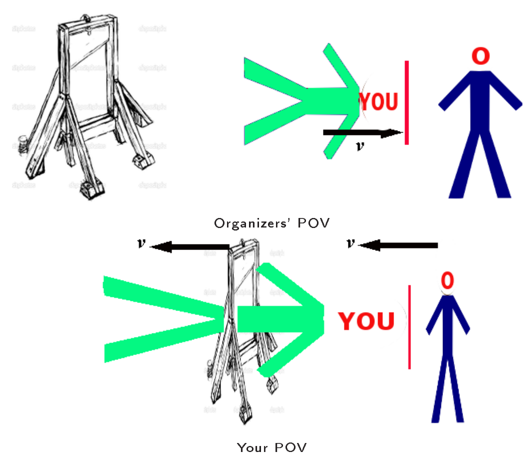


Simultaneous arrival of the signals.

28

(21) You're 6 feet tall and are diving (at high speed) on a race course toward a finish line equipped with a sensor. The moment you cross the line, a blade falls six feet behind the finish line. The organizers say that you'll be length-contracted as you dive, and your feet will be well past where the blade falls. Your mother, taking your point of view, says that the race course will be contracted relative to you and the blade will fall on you. Who's right?

29



30

ADDITIONAL NOTES

(22) The twin paradox: Consider two twins, one a collective, one singular (same total age):

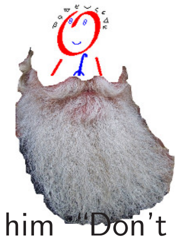
COLLECTIVE
YOU

The twin-collective sets off on a journey. . .

31

The right twin says that they are moving so they should return younger than he:

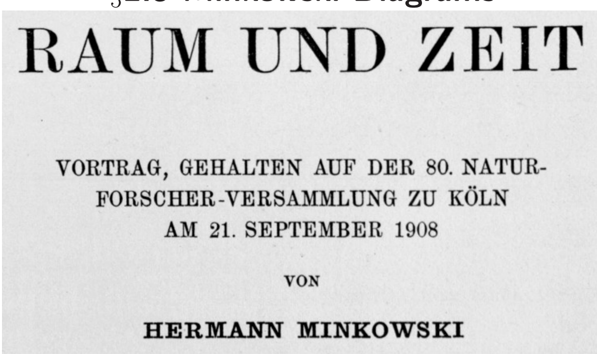
COLLECTIVE
YOU



But the twin-collective says to him "Don't you know any relativity? Relative to us, you've moved, so you're younger" . . . Who's right, and why?

32

§2.5 Minkowski Diagrams



RAUM UND ZEIT

VORTRAG, GEHALTEN AUF DER 80. NATURFORSCHER-VERSAMMLUNG ZU KÖLN
AM 21. SEPTEMBER 1908

VON

HERMANN MINKOWSKI

"Space and Time"

33

Minkowski said:

> Gentlemen! The views of space and time which I want to present to you arose from the domain of experimental physics, and therein lies their strength. Their tendency is radical. From now onwards space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two will still stand independently on its own.

Lecture delivered before the Naturforscher Versammlung (Congress of Natural Philosophers) at Cologne, September 21, 1908.

34

In this paper, Minkowski introduced diagrams, we now name after him, which represent the profound idea of ____________, and, through it, a visual, geometrical approach to relativity.

35

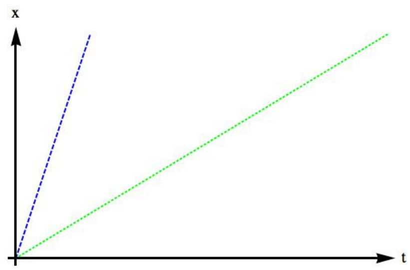
Although Einstein's theory of Special Relativity had received immediate acceptance, it was this paper by Minkowski that guided future research.

Einstein initially scoffed at what he thought was an overly mathematical approach to his theory but he came to realize that it did represent an important step forward.

36

ADDITIONAL NOTES

Consider two objects moving in one space dimension, and plot their positions as a function of time:



37

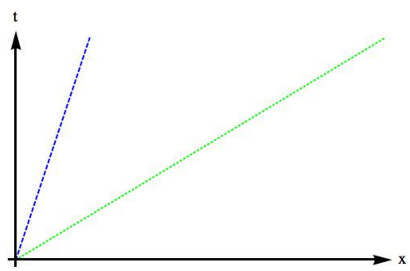
(23) Which of the two plots represents a higher speed? ______________________

____

(24) What geometrical aspect of the previous plots represents the speed? ________________

38

In the diagrams that Minkowski introduced, the time axis is represented vertically:



39

(25) *Now*, which of the two plots represents a higher speed? ______________________

____

____

(26) What geometrical aspect of the previous plots represents the speed? _______________________

40

§2.6 Spacetime

Minkowski unified space and time into spacetime.

An ____________________________ is an entity (aka $\mathbb{R}^n$) with coordinates $(t, x_1, x_2, x_3, \ldots, x_{n-1})$.

You may think of t as "time" and $x_1, x_2, \ldots x_{n-1}$ as "space."

41

We'll mostly look at $4d$ space-time with coordinates such as (t, x, y, z) or (t, x_1, x_2, x_3). We'll use other coordinates (such as polar) as needed.

In order to be able to draw pictures, many of our examples will be drawn in 2d [coordinates (t, x)] or 3d [coordinates (t, x, y)].

These pictures are drawn with the t axis vertical.

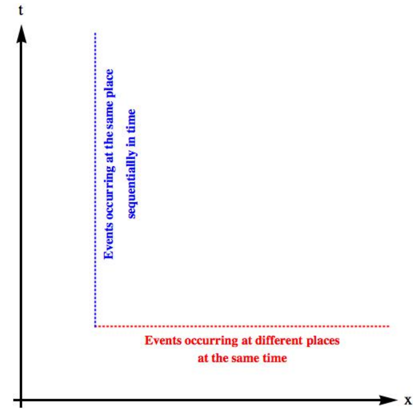
42

ADDITIONAL NOTES

A single spacetime point, p, is an __________. It's ______________________________ ______________________________.

A curve in spacetime is a ______________. Examples include (a) events that may be thought of as occurring sequentially in time, or (b) events that may be thought of as occurring at different places at the same time.

43



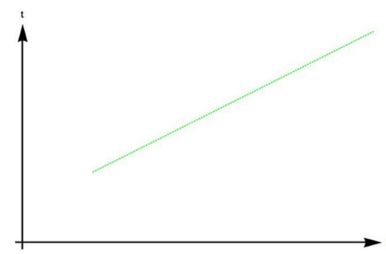
44

§2.7 Timelike and Spacelike Lines

The vertical curve on the previous diagram is called ___________ and it represents events that occur sequentially in time (they are in ______ _____________ because prior events can send signals to or influence later ones).

The horizontal curve is called ____________ and it represents ____________________.

45



What about a curve that's neither vertical nor horizontal such as this one? Is it "timelike" or "spacelike"? Does the question even make sense?

46

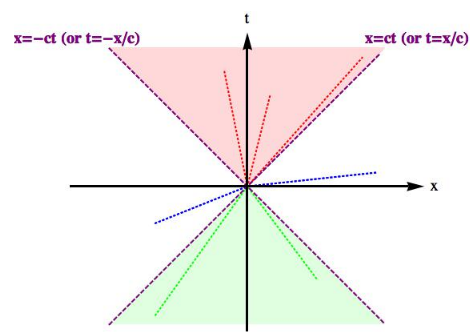
§2.8 Light Cones

That's where _____________ help.

____ ____.

The straight lines emanating from P on the light cone are called ___________ or _________.

47



Null lines from the origin have equations ______.

48

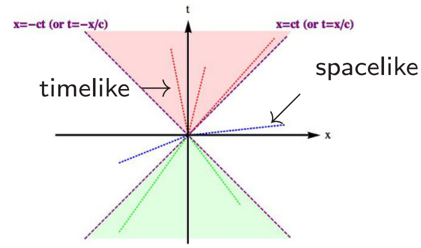
ADDITIONAL NOTES

(27) What is their slope? ______________

Straight lines from P _________________________ (e.g., within the shaded regions in the diagram) are called ________________.

49

Straight lines from P outside the light cone are called ________________.



50

The |spacetime slope| criteria for a straightline segment are:

timelike: ______________

spacelike: ______________

lightlike: ______________

where "spacetime slope" in 2d is _______________.

51

Relative to the speed of light, c, what speed would you have to be traveling to traverse

(28) a lightlike (null) line? _________.

(29) a timelike line? ____________.

(30) a spacelike line? ____________.

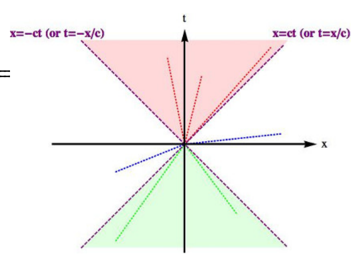
52

(31) Why are there two light cones at P?

____

and

____



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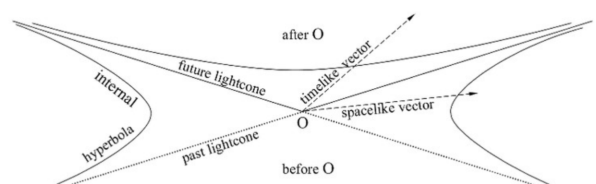


Fig. 2

Minkowski's terms were:

Vorkegel – "in front cone" (past cone)

Nachkegel – "after cone" (future cone)

54

ADDITIONAL NOTES

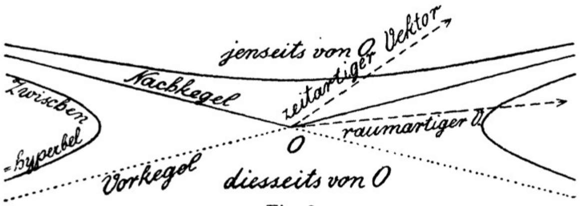
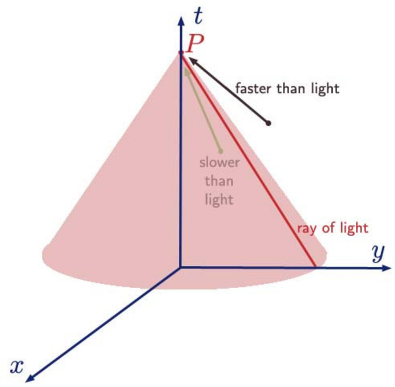


Fig. 2.

55



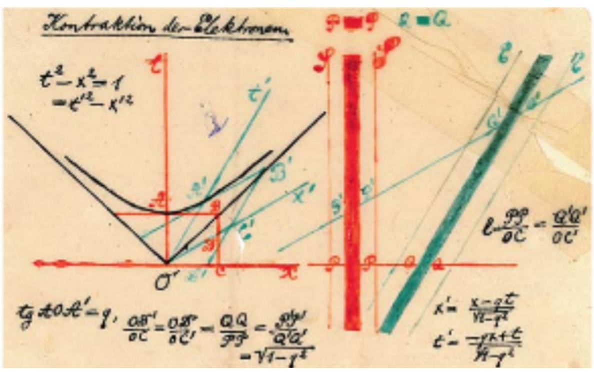
Why "light cones"?

____

____

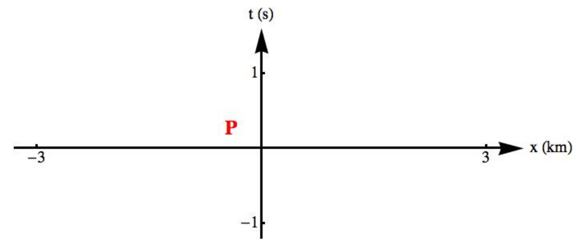
____

56



57

(32) $c \approx 3 \times 10^5$ km/s. What would the light cone of point P look like on a graph that has distance plotted between ± 3 km and time between ± 1 sec?



58

(33) If t is measured in hundred-thousandths of a second and x in km, what value would c have?

____

In these units classify the line segments between these points as timelike, spacelike or null:

(34) $(1{,}5)$ and $(2{,}7)$: ____________________

(35) $(-1{,}4)$ and $(1{,}-5)$: ____________________

(36) $(2{,}1)$ and $(3{,}4)$: _____________

59

§2.9 Invariances

If we label our coordinates as $(x_1, x_2, \ldots x_n)$ and use the notation that Δx is the difference in x, then we can think of the Pythagorean distance formula as

____

This is __________________________________

60

ADDITIONAL NOTES

What does this mean? If we transform to new coordinates moving at velocity $\vec{v} = (v_1, v_2, \ldots v_n)$ with respect to the old, we'll have

61

"Invariance" means both coordinate systems give the same Pythagorean distance.
(37) Show this.

.

62

This means that two coordinate systems that are moving uniformly relative to each other will measure the same distances/sizes, *as long as Galilean transformations are the correct ones to use*.

But we know they are not the correct transformations, and that lengths are not invariant.

63

What about the Lorentz Transformations?

$$\hat{t} = \gamma(t - vx/c^2) \qquad \hat{x} = \gamma(x - vt)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Therefore

$$\Delta\hat{t} = \underline{\hspace{6cm}}$$

$$\Delta\hat{x} = \underline{\hspace{5cm}}$$

64

The Pythagorean distance is not invariant under these transformations. But, ~~we~~ you can
(38) show that $c^2\Delta t^2 - \Delta x^2$ is.

65

66

ADDITIONAL NOTES

The preceding is analogous to the invariance of the Pythagorean distance under Galilean transformations.

The quantity $c^2\Delta t^2 - \Delta x^2$ is very important. It is called the squared ________________. This concept will be with us for the rest of the course, from now to the beginning (of the Universe).

67

§2.10 Proper Time

The squared ________________ between two points is given by

____

We've seen (in 2d at least) that this is invariant under Lorentz transformations.

68 The ________________ is $\sqrt{\Delta s^2}/c = \Delta s/c$.

With $c = 3$, as before, what is the squared proper interval between each of these?

(39) $(1, 5)$ and $(2, 7)$.

______________________________.

(40) $(-1, 4)$ and $(1, -5)$

______________________________.

(41) $(2, 1)$ and $(3, 4)$

______________________________.

69

(42) How do we get a proper time out of these answers, and what are its units? __________

____

(43) What connection do you guess between type of line (timelike, null, etc.) and proper time?

____

____

____

70

When an observer follows a spacetime curve, that curve is called his/her ______________. The proper time along a worldline is the actual time experienced and measured by the observer on it.

This concept can be used to clarify several "paradoxes," such as the twin paradox.

71

(44) If $c = 3 \times 10^5$ km/s, how many km does light travel in a year?

____

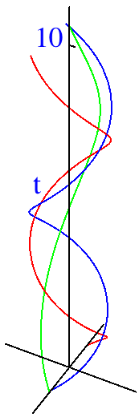
(45) What's that in the language of gazillions?

____

(46) Another name for this? ______________

72 **(47)** What's c in units of ly and y? __________

ADDITIONAL NOTES



The different worldlines shown on the left will "experience" different physical times even though they all start at $t = 0$ and stop at $t = 10$.

The physical time is obtained by calculating the proper time along each curve.

73

§2.11 Geometry of the Lorentz Transformations

$$\hat{t} = \gamma(t - vx/c^2)$$
$$\hat{x} = \gamma(x - vt)$$
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

74

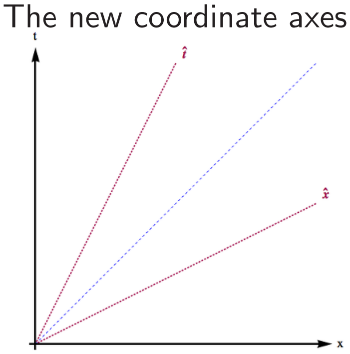
What do the $\hat{x}$ and $\hat{t}$ coordinate lines look like in the x-t coordinate system? To be concrete, suppose we use units where $c = 1$ and $v = 1/2$. Then

$\hat{t}$ axis: ______________________________

$\hat{x}$ axis: ________________________________

75

The new coordinate axes look like this:

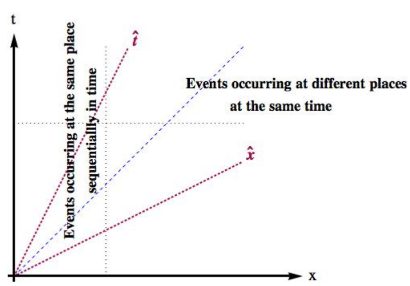


Slope of $\hat{x}$ axis: v.
Slope of $\hat{t}$ axis: $1/v$
(units where $c = 1$).

Can also study this dynamically.

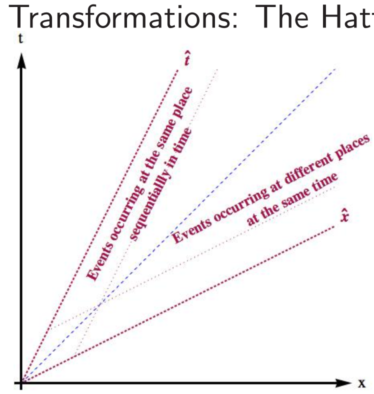
76

Lorentz Transformations: The Unhatted POV



77

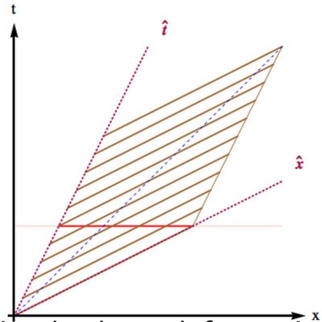
Lorentz Transformations: The Hatted POV



78

ADDITIONAL NOTES

Length Contraction:



Rod at rest in the hatted frame is seen as contracted from the unhatted frame.

79

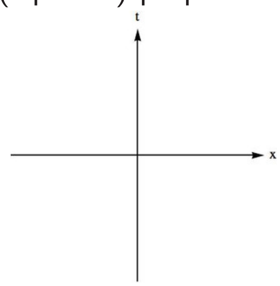
§2.12 More on Proper Time

(48) In 2d (1 space, 1 time), if $(t_1, x_1) = (0, 0)$ (aka "the origin") and $(t_2, x_2) = (t, x)$ what are Δt and Δx? ______________________.

(49) If $c = 1$, what is the squared proper time interval between $(0, 0)$ and (t, x)?

80

(50) In 2d spacetime, with $c = 1$, plot the null lines through $(0, 0)$, the line with $t = 1$, and the line with that (squared) proper time equal to 1.



81

What do you recognize in Minkowski's diagram?

$$c^2t^2 - x^2 - y^2 - z^2 = 1.$$

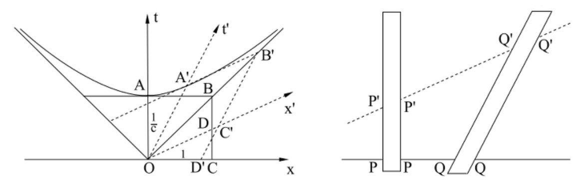


Fig. 1

82

§2.13 Matrix Language

Because both the Galilean and Lorentz transformations are linear, they can be expressed via matrices.

83

The Galilean World

If a system is moving with velocity $\vec{v} = (v_1, v_2, v_3)$, the transformation from a "stationary" system is

$$\begin{pmatrix} \hat{t} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ -v_2 & 0 & 1 & 0 \\ -v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

84

ADDITIONAL NOTES

(51) Multiply the matrices and write this as a set of equations.

85

The Lorentz-Minkowski World

If a hatted system is moving with respect to another, the 2d (and nd) Lorentz transformation can also be expressed in matrix form:

$$\begin{pmatrix} \hat{t} \\ \hat{x} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v/c^2 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$
$$= \gamma \begin{pmatrix} t - vx/c^2 \\ -vt + x \end{pmatrix}$$

86

§2.14 The Metric

The metric is a special matrix that allows you to find "distances." For example, distances on

$3d$ space in Cartesian coordinates:

$$ds^2 = dx^2 + dy^2 + dz^2$$

2d sphere of radius r:

$$ds^2 = r^2 d\theta^2 + r^2\, sin^2\, \theta d\phi^2$$

87

These formulas are examples of a general purpose formula for an n-dimensional distance

$$ds^2 = \sum_{i,j}^{n} g_{ij} dx^i dx^j$$
$$= g_{11} dx^1 dx^2 + g_{12} dx^1 dx^2 \ldots$$

where dx^i is the differential (small difference) in the ith coordinate, *not a power*.

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The expression $\sum_{i,j}^{n} g_{ij} dx^i dx^j$ can be calculated using matrix multiplication, *provided you write it out in a specific form*:

$$(dx^1, dx^2, \ldots) \begin{pmatrix} g_{11} & g_{12} & \cdots \\ g_{21} & g_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ \vdots \end{pmatrix}$$

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The expression $\sum_{i,j}^{n} g_{ij} dx^i dx^j$ occurs so often that the summation is often taken as implied, and it's written simply as

$$g_{ij} dx^i dx^j$$

(the "Einstein summation convention.")

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ADDITIONAL NOTES

3d Pythagorean metric

$$ds^2 = (dx^1, dx^2, dx^3)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}dx^1\\ dx^2\\ dx^3\end{pmatrix}$$

91

(52) Multiply and verify.

92

Minkowski metric:

$ds^2 =$

$$(cdt, dx, dy, dz)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}cdt\\ dx\\ dy\\ dz\end{pmatrix}$$

93

(53) Multiply and verify.

94

Both the Pythagorean and Minkowski metrics are constant.

Metrics such as these are called ________, because the curvatures that can be computed from them are zero.

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Christoffel symbols (second kind):

$$\Gamma^m_{ij} = g^{mk}\Gamma_{ijk} = \frac{1}{2}g^{mk}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^l} - \frac{\partial g_{ij}}{\partial u^k}\right)$$

Reimann-Christoffel curvature tensor:

$$R^m_{ikj} \equiv \frac{\partial \Gamma^m_{ij}}{\partial u^k} - \frac{\partial \Gamma^m_{ik}}{\partial u^j} + \Gamma^n_{ij}\Gamma^m_{nk} - \Gamma^n_{ik}\Gamma^m_{nj}$$

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ADDITIONAL NOTES

§2.15 Spacetime: A Summary

Special relativity is clarified by Minkowski's spacetime: a 4-d (or n-d) entity unifying space and time.

If spacetime slope is $|\Delta t/\Delta x|$, we classify

Slope $> 1/c$: ________________________

Slope $< 1/c$: ______________________

Slope $= 1/c$: _____________________

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The **proper time** is $\Delta s/c$ (Δs is proper interval) and is the physical time measured along an observer's worldline.

$\Delta s^2 > 0$: ________________________

$\Delta s^2 < 0$: ______________________

$\Delta s^2 = 0$: _____________________

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§2.16 Resolving "Paradoxes"

(54) *Einstein's Train*: A train passes by you with a guard in the middle. Just as the guard passes you, flashes of light emitted from the front and the back of the train reach both you and the guard. Both of you agree on this. Do you agree on when the flashes were emitted?

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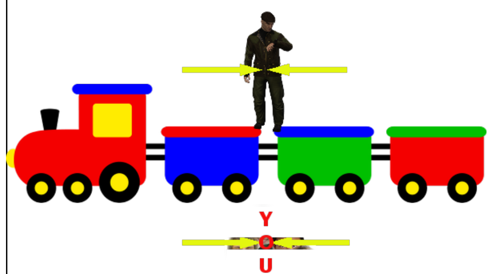

Simultaneous arrival of the signals.

100

Let the guard's frame be "unhatted," and the "rest length" of the train be 2ℓ.

Let $t = 0 = \hat{t}$ be the instant when both you and the guard cross.

In the guard's frame the signals are emitted at

$$x_f = -\ell, \quad x_b = \ell, \qquad t = -\ell/c$$

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Apply the LTs:

$$\hat{t} = \gamma(t - vx/c^2), \qquad \hat{x} = \gamma(x - vt)$$

In your frame:

$$\hat{t}_b = \gamma\left(\frac{-\ell}{c} - \frac{v\ell}{c^2}\right) = -\frac{\gamma}{c}\left(1 + \frac{v}{c}\right)\ell$$

$$\hat{t}_f = \gamma\left(\frac{-\ell}{c} + \frac{v\ell}{c^2}\right) = -\frac{\gamma}{c}\left(1 - \frac{v}{c}\right)\ell$$

So $\hat{t}_b < \hat{t}_f$.

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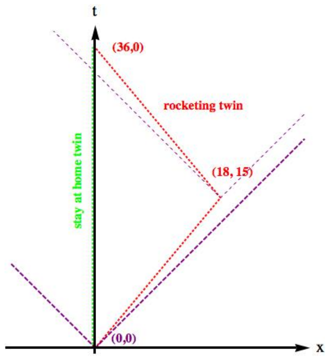
ADDITIONAL NOTES

Twin paradox:

Here's a sketch of a particular "twin paradox situation."

Assume, distance is in l-y and time in y, so $c = 1$



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(55) From the diagram on the previous slide, calculate the proper times for both the stay at home twin and the rocketing twin, *from the earth pov.*

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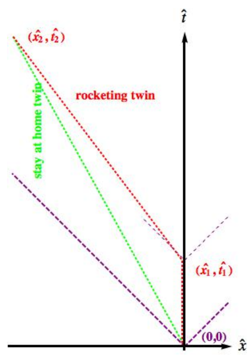
(56) Do the same from the pov of a frame attached to the outgoing twin.



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Rocket time:

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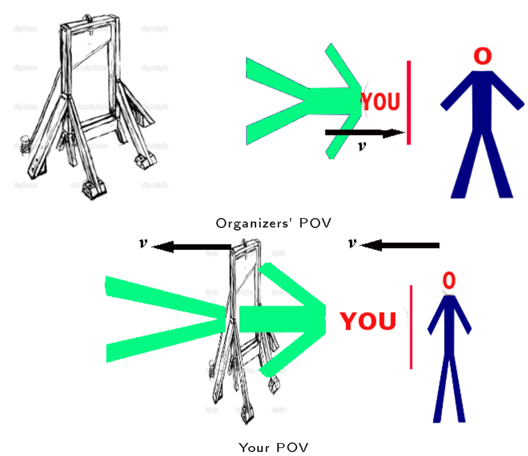

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ADDITIONAL NOTES