## §9.1 Driven Damped Oscillations

Oscillations typically die. But, we can stimulate them into staying alive by supplying an external force to drive them:

**(1)** Divide this by $m$, and bring in our old friends $\omega_0 = \sqrt{k/m}$, and $2\beta = b/m$:

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Let's introduce an "operator" D, where

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2.$$

What this means is that

$$Dx = \frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x.$$

Then the equation for driven damped oscillations is simply $Dx(t) = f(t)$.

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**(2)** Drawing on your vast prev. calc. knowledge,

1. $(f_1 + f_2)' =$ ____________
2. $(af)' =$ ________
3. Etc.

It follows that

$$D(a_1x_1 + a_2x_2) = a_1Dx_1 + a_2Dx_2.$$

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Suppose we have a particular solution of the de for a driven damped oscillator that we call $x_p$. In other words

$$Dx_p = f.$$

Suppose we have a solution, $x_h$, of the homogeneous equation, $Dx_h = 0$.

**(3)** What is $D(x_p + x_h)$?

$D(x_p + x_h) =$

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So, to find the general solution of $Dx = f$, we need to find a particular solution, $x_p$, and a general homogenous solution that obeys $Dx_h = 0$.

Then $x_p + x_h$ will be the general solution of the de of interest.

The general solution will have two constants of integration.

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We already know that

$$x_h(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

is the general solution of the homogenous eqn.

But we need to know something further to get a (particular) solution of

$$Dx = \frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = f.$$

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ADDITIONAL NOTES

(4) What's that? ________
Without knowing what $f(t)$, the driving force, is we can go no further.

If we're interested in driving oscillations let's examine an oscillatory ("sinusoidal") driving force:

$$f(t) = f_0\, cos(\omega t).$$

(The $\omega$ here is not the "natural frequency," $\omega_0$.)

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Then

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\, cos(\omega t).$$

But, $cos$ and $sin$ differ only by phase, so we have

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0\, sin(\omega t).$$

So, if $z = x + iy$, we get,

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}.$$

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So, we hunt for a function, $z(t)$, that has the property that $z$, $\dot{z}$ and $\ddot{z}$ are all proportional to $z$.
(5) What *might* $z$ be (or proportional to)?

If that's true, we have

$$z =$$

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(6) So, what's $C$?

$$C = \rule{4cm}{0.4pt}$$

$C$ is itself complex. If we write it as

$$C =$$

it follows after significant algebra, that

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$$A =$$

and

$$\delta =$$

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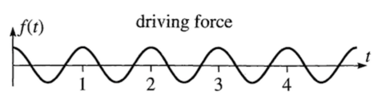
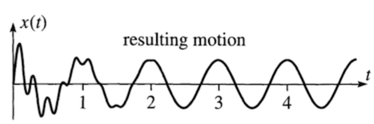
The full solution is

$$x(t) =$$



(a)



(b)

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ADDITIONAL NOTES

In other words, as we saw earlier, the

$$x_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

part of the solution dies out and we're left with

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## §9.2 Resonance

$$A = \sqrt{\frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

(7) What's $A$ for $\beta \approx 0$?

$$A =$$

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(8) As the driven frequency approaches the natural, $\omega \to \omega_0$, the amplitude approaches,

$$A \to \infty.$$

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ADDITIONAL NOTES

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