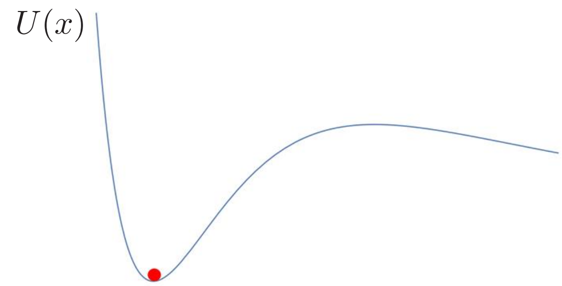
Arvind Borde / PHY 17, Week 8: Oscillations

§8.1 Introduction

An oscillation is a back-and-forth motion around an equilibrium position. The equilibrium position has the property that if the system is initially there, with no initial velocity, it will stay there.

Such oscillations can happen in any number of dimensions, and can occur whenever there is a "po-
1 tential energy well."

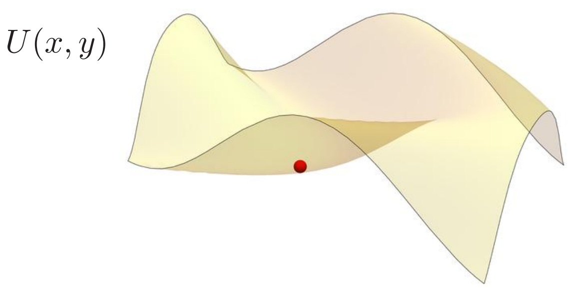
§8.1.1 Particle in a 1-d potential well



Equilibrium position.

2

§8.1.2 Particle in a 2-d potential well



Particle at equilibrium in a 2-d well. Here, too, small devi-
3 ations from equil. give oscillations, large ones do not.

§8.2 Simple Harmonic Motion

SHM is a particularly simple kind of oscillation that arises from a potential energy of the form

$$U = \frac{1}{2}kx^2,$$

where k is a constant and x is the distance from equilibrium. Such a potential is the simplest (poly-
4 nomial) that has a well.

No realistic potential has exactly this form, but it is nevertheless an excellent approximation to the truth when the oscillations are small, which (from above) they usually are.

Suppose (in 1-d) that $U(x)$ is a potential with wells, and that we're interested in oscillations in one particular well. Pick the location of the bot-
5 tom of that well to be $x = 0$, and pick $U(0) = 0$.

(1) What's the Taylor series expansion for $U(x)$ around $x = 0$?

$U(x) =$

,

(2) Why did the first two terms vanish?

6 ____

____

ADDITIONAL NOTES

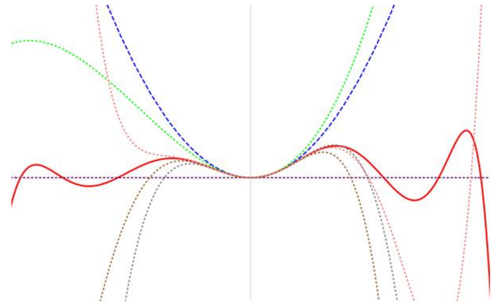
For small x, we approximate by retaining only the leading term

$$U(x) \approx \frac{1}{2}U''(0)x^2 \equiv \frac{1}{2}kx^2.$$

This is what makes understanding SHM, simple though it is (by its own name – you ain't called "Simple Simon" if you ain't simple), important.

7 It's useful to visualize this, as well.

The first few terms in the Taylor expansion of a hypothetical $U(x)$ (solid line) with a well at $x = 0$.



8

So, to first non-vanishing order, we can approximate any potential well as

$$U(x) = \frac{1}{2}kx^2,$$

where $k = U''(0)$, as long as we choose (as we always can) $x = 0$ as the location of the well, and the zero of PE (equivalently, the reference point wrt which we calculate PE) such that $U(0) = 0$.

9

(3) I sense something worries you. What is it, you worry-warriers?

10

(4) Why does the sign of k matter?

____

11

If $k < 0$, no well, so no oscillations.
(Equilibrium at top of hill is unstable.)

(5) For a general potential, are we guaranteed that $k = U''(0) > 0$?

____

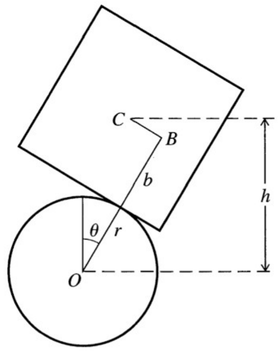
____

____

12 $U''(0)$

ADDITIONAL NOTES

§8.2.1 Example: block on a cylinder



13

Scenario:

- Cylinder has radius r.
- Block has side $2b$.
- Block is initially placed with its center above cylinder center, then perturbed slightly.
- Block rolls without slipping.
- Block has mass m and uniform density.

14

Under these conditions, the CofM of the block is its geometrical center. If we take the zero of grav. PE as the height of the center of the cylinder, the PE of the block will be mgh, where h is as shown in the diagram.

Let's pick θ, as shown in the diagram, as our one relevant variable.

15

Observations:

- Distance point of contact moves: $r\theta = BC$.
- Angle between BC and horizontal: θ.
- *Height* of C above B: $BC\, sin\,\theta = r\theta\, sin\,\theta$.
- Height of B above ref. level: $(b + r)\, cos\,\theta$.

So h, the height of C above the ref. level, is

$$h =$$

16

So $U(\theta) = mg\,[(b + r)\, cos\,\theta + r\theta\, sin\,\theta]$.

(6) What's $U'(0)$?

$U'(\theta) =$

17

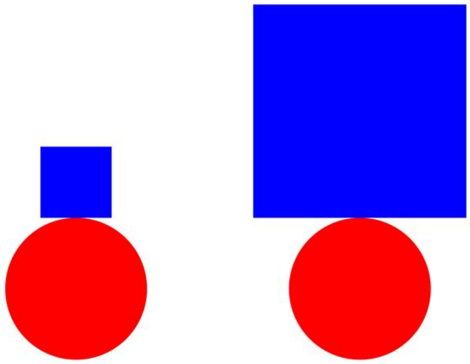
(7) Is this equilibrium stable ($U''(0) > 0$), or unstable ($U''(0) < 0$)?

$U''(\theta) =$

(8) Makes sense? ________

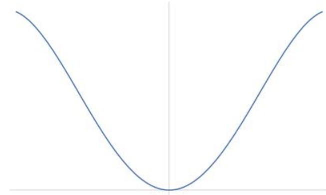
18

ADDITIONAL NOTES



19

Although we have a condition for stable equilibrium, it doesn't follow that the motion of a small cube on a large cylinder will be SHM. If we pick $r = 1$ unit and $b = 0.5$ unit and plot U we get a graph that's not a parabola:



20

But for small oscillations (i.e., small θ) we can use

$$cos\,\theta \approx 1 - \theta^2/2, \qquad sin\,\theta \approx \theta.$$

(9) What's $U(\theta)$ for small θ?

$U(\theta) \approx$

The first term is constant and can be eliminated by shifting the "zero" of PE. The second term has the SHM form, with $k = mg(r - b) = U''(0)$.

21

§8.2.2 Other examples

A mass oscillating on a spring, vertically or horizontally, a pendulum, etc., are other examples of systems which obey the equation for SHM to a high degree of accuracy when the oscillations are small.

22

§8.3 Solving the SHM Equation

The force associated with pot. $U(x) = {}^1\!/_2 kx^2$ is

$$F(x) = -\frac{dU}{dx} = -kx.$$

By NSL, we can write this as

$$\ddot{x} = -(k/m)x \equiv -\omega^2 x,$$

where ω is defined to be $\sqrt{k/m}$.

23

We've already discussed functions whose second derivatives are proportional to the negatives of themselves, and have gotten

$$x(t) = B_1\,cos(\omega t) + B_2\,sin(\omega t)$$

as the general solution.

24

ADDITIONAL NOTES

(10) Find B_1 and B_2 in terms of x_0 and v_0, the initial position and velocity, respectively.

So we can think of the general solution as

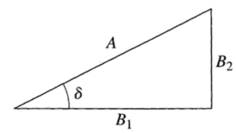
$$x(t) = x_0\, cos(\omega t) + (v_0/\omega)\, sin(\omega t).$$

A solution with $x_0 = 0$ is $x(t) = (v_0/\omega)\, sin(\omega t)$,
25 and one with $v_0 = 0$ is $x(t) = x_0\, cos(\omega t)$.

(11) What's $x(t)$ if x_0 and v_0 are both zero?

____

The more interesting case is when x_0 and v_0 are both nonzero. Introduce $A^2 = B_1^2 + B_2^2$, and think of the three quantities as sides of an abstract triangle:



26

Writing the solution as

$$x(t) = A\left[\frac{B_1}{A}\, cos(\omega t) + \frac{B_2}{A}\, sin(\omega t)\right]$$

we get from the geometry of the diagram

$$x(t) = A\,[\qquad\qquad\qquad\qquad\qquad\qquad]$$
$$= A$$

27 We call δ the ____________________.

This form of the solution $x(t)$ is called the phase-shifted form. Different values of the phase, δ, give us different possibilities for x_0. We can also get the velocity in this form

$$v(t) \equiv \dot{x}(t) =$$

28

§8.4 Energy

Substituting the previous solution for $x(t)$ into the PE $U(x)$, we have

$$U(x(t)) = \tfrac{1}{2}kx^2(t) =$$

And substituting $v(t)$ into the KE, we have

$$\tfrac{1}{2}mv^2(t) =$$

29

(12) Show that the total mechanical energy E (KE + PE) has a fixed value for all values of t (i.e.,is conserved). What is that value of E?

____

$E =$

30 This establishes the constancy of E and its value.

ADDITIONAL NOTES

§8.4.1 The interplay between KE and PE

KE: $\frac{1}{2}mv^2(t) = \frac{1}{2}mA^2\omega^2\, sin^2(\omega t - \delta)$.

PE : $\frac{1}{2}kx^2(t) = \frac{1}{2}kA^2\, cos^2(\omega t - \delta)$.

Because the cos and sin functions are perfectly out of phase, so are KE and PE: when one peaks the other is zero; as one goes down, the other goes up. That's what allows E to be conserved.

31

§8.5 Damped Oscillations

Nothing in the real world is forever, not a pendulum, not a mass bobbing on a spring, not even a cube rolling on a cylinder. Drag/friction afflicts them all. Assuming drag forces are proportional to velocity, a realistic oscillator is likely to obey NSL in this form:

$$m\ddot{x} = -kx +$$

32

Defining $\omega_0 = \sqrt{k/m}$ (the subscript distinguishes this frequency, called the *natural frequency* of the system, from others that may arise) and $\beta = b/2m$, called the damping constant, NSL becomes

$$= 0.$$

DEs of this type can be solved by using $x = e^{rt}$ as a trial balloon.

33

(13) Plug this into our eqn.: what do you get?

So we've replaced a de with an algebraic, quadratic equation. The solutions are

$$r_\pm =$$

Let $r_1 = r_+$, and $r_2 = r_-$.

34

The general solution to the de is then

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$
$$=$$

To understand this better, let's look at subcases.

35

§8.5.1 Subcase 1: the undamped ($\beta = 0$)

$x(t) =$

36

ADDITIONAL NOTES

§8.5.2 Subcase 2: the weakly damped ($\beta < \omega_0$)

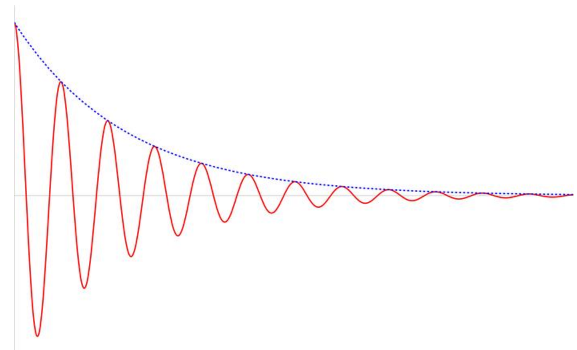
Define $\widehat{\omega} = \sqrt{\omega_0^2 - \beta^2}$. ($\widehat{\omega}$ is real.)

Then $\sqrt{\beta^2 - \omega_0^2} = i\widehat{\omega}$, and the solution is

$$x(t) =$$

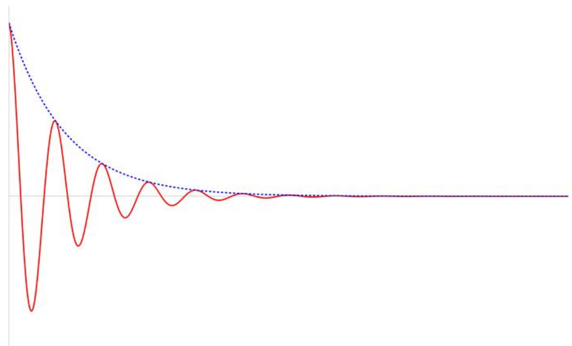
37

Weakly damped ("underdamped") oscillations:



38

Doubling β:



39

§8.5.3 Subcase 3: the strongly damped ($\beta > \omega_0$)

The solution is

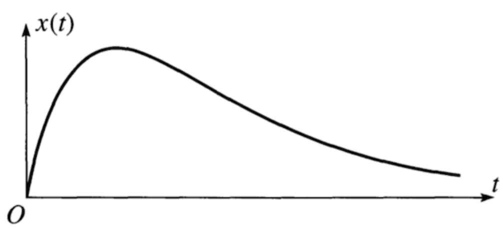
$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}$$

The left-hand exponential decays. The right-hand one does, too.

(14) Why?

40

The oscillations are killed here ("overdamped"):



41

§8.5.4 Subcase 4: the critically damped ($\beta = \omega_0$)

Looking at the overdamped case, we see that

$$x(t) =$$

That's just one solution, and we need two for a second-order de. Going back to the drawing board (slide 33), in this special case, NSL is

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0.$$

42

ADDITIONAL NOTES

(15) Check $x(t) = te^{-\beta t}$ as a possible solution.

$\dot{x}(t) =$

$\ddot{x}(t) =$

So $\ddot{x} + 2\beta\dot{x} + \beta^2 x$ is

43

The general critically damped solution is

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}.$$

This decays the quickest of all damped oscillations, and getting close to it is desirable for shock-absorbers and suchlike.

44

§8.6 Oscillations in 2-d

NSL in the case of independent oscillations of a particle of mass m in the x and y directions is

$$\ddot{x} = -\omega_x^2\, x, \qquad \omega_x = \sqrt{k_x/m}$$
$$\ddot{y} = -\omega_y^2\, y, \qquad \omega_y = \sqrt{k_y/m}$$

45

The solutions are

$$x(t) = A_x\, cos(\omega_x t - \delta_x)$$
$$x(t) = A_y\, cos(\omega_y t - \delta_y)$$

which may be re-written as

$$x(t) = A_x\, cos(\omega_x t)$$
$$x(t) = A_y\, cos(\omega_y t - \delta),$$

46 where $\delta = \delta_y - \delta_x$.

The solutions represent paths that are periodic separately in x and y, but may not be in the x-y plane.

47

ADDITIONAL NOTES