§2.1 Newton's Laws & Momentum Conservation

§2.1.1 Momentum conservation: single particle

"The momentum, $\vec{p}$, of a particle is constant if the total external force acting on it is zero."

This follows from one of Newton's laws.

(1) Which? ____________

(2) How? ________________

____

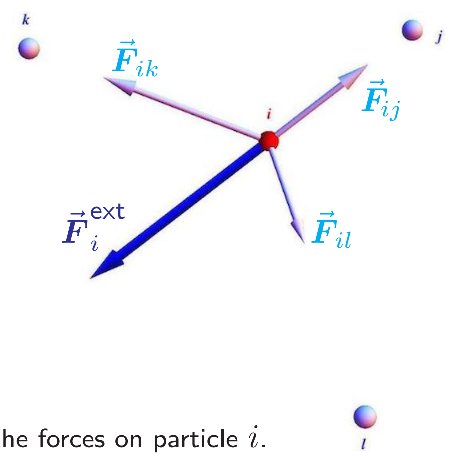
1 ________________________________

§2.1.2 Multiparticle systems: forces

In a system of N particles, every particle has two sets of forces acting on it, external and internal. If $\vec{F}_{ij}$ represents the internal force exerted by the j-th particle on the i-th, we have for particle i:

1) $\displaystyle\sum_{j=1 (j\neq i)}^{N} \vec{F}_{ij}$

2 2) $\vec{F}_i^{\,\text{ext}}$, the *net* external force on particle i.



3 Some of the forces on particle i.

Total force on particle i:

$$\vec{F}_i = \sum_{j=1 (j\neq i)}^{N} \vec{F}_{ij} + \vec{F}_i^{\,\text{ext}}.$$

Total force on the system:

$$\vec{F}^{\,\text{tot}} = \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} \left[\sum_{j=1 (j\neq i)}^{N} \vec{F}_{ij} + \vec{F}_i^{\,\text{ext}} \right].$$

4

We may rewrite this as

$$\vec{F}^{\,\text{tot}} = \sum_{i=1}^{N} \left[\sum_{j=1 (j\neq i)}^{N} \vec{F}_{ij} \right] + \sum_{i=1}^{N} \vec{F}_i^{\,\text{ext}}.$$

(3) Why is this OK?

____

5

(4) Write out a few explicit terms from

$$\sum_{i=1}^{N} \left[\sum_{j=1 (j\neq i)}^{N} \vec{F}_{ij} \right].$$

$i = 1$:

$i = 2$:

6 $i = 3$:

ADDITIONAL NOTES

What we've shown formally is the intuitively believable result that when you calculate the net force on a system of particles, the internal forces cancel, because – from NTL – they come in "equal and opposite pairs:"

$$\sum_{i=1}^{N}\left[\sum_{j=1(j\neq i)}^{N}\vec{\boldsymbol{F}}_{ij}\right]=\vec{\mathbf{0}}.$$

7

Therefore, the total force on a system of particles is the sum of the net external forces on each particle:

$$\vec{\boldsymbol{F}}^{\text{tot}}=\sum_{i=1}^{N}\vec{\boldsymbol{F}}_i^{\text{ext}}.$$

8

§2.1.3 Multiparticle systems: momentum

Since $\vec{\boldsymbol{F}}=m\vec{\boldsymbol{a}}=m\dot{\vec{\boldsymbol{v}}}=(\dot{m\vec{\boldsymbol{v}}})$, the momentum of a particle obeys

$$\vec{\boldsymbol{F}}=\dot{\vec{\boldsymbol{p}}}.$$

So, the momentum, $\vec{\boldsymbol{p}}_i$, of particle i in a system of N particles obeys

$$\dot{\vec{\boldsymbol{p}}}_i=\vec{\boldsymbol{F}}_i=\sum_{j=1(j\neq i)}^{N}\vec{\boldsymbol{F}}_{ij}+\vec{\boldsymbol{F}}_i^{\text{ext}}.$$

9

Total momentum of a system of N particles:

$$\vec{\boldsymbol{P}}=\sum_{i=1}^{N}\vec{\boldsymbol{p}}_i.$$

Therefore,

$$\dot{\vec{\boldsymbol{P}}}=\sum_{i=1}^{N}\dot{\vec{\boldsymbol{p}}}_i.$$

10

§2.1.4 Momentum conservation: multiparticles

"*The total momentum, $\vec{\boldsymbol{P}}$, of a multiparticle system is constant if the total external force acting on it (the system) is zero.*"

Proof:

$$\dot{\vec{\boldsymbol{P}}}=\sum_{i=1}^{N}\dot{\vec{\boldsymbol{p}}}_i=\sum_{i=1}^{N}\left[\sum_{j=1(j\neq i)}^{N}\vec{\boldsymbol{F}}_{ij}+\vec{\boldsymbol{F}}_i^{\text{ext}}\right]=\sum_{i=1}^{N}\vec{\boldsymbol{F}}_i^{\text{ext}}.$$

11

§2.2 Momentum Conservation

Newton's Laws are the fundamental laws of classical mechanics. We've derived momentum conservation as a consequence.

It turns out the conservation of momentum is so fundamental that it holds even when classical mechanics does not – such as in the relativistic domain (high speeds), or the quantum (small sizes).

12

ADDITIONAL NOTES

§2.3 Particles vs Fields

Newton grappled with the question of *how* particles separated by a distance exerted forces on each other (such as the gravitational force). He seemed to come down on the side of instantaneous "action at a distance."

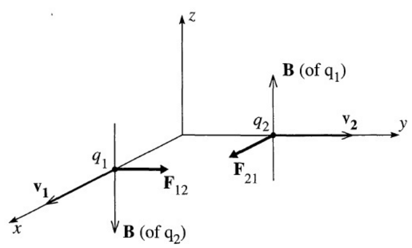
We now "know" that forces are "interactions" carried by fields at finite speeds.

13

Without fields, we are in peril (or, at least, our understanding of the world is).

Consider two positive charges, q_1 and q_2, moving in the positive x and y directions, respectively. The magnetic forces they exert on each other are equal in magnitude, but *not* opposite in direction.

14



Fields and forces from cross-products (RH rules).

It follows that the total *mechanical* momentum of the particles, $m_1\vec{v}_1 + m_2\vec{v}_2$, is not conserved.

15

(5) What gives? ______________________

Also, in the overall picture, there's an electric force between the particles that obeys NTL. At low speeds it dominates the magnetic. If the particles are separated by distance d, the magnitude of the electric force is

$$F_{12}^{\text{elec}} =$$

16

The magnetic field created by, say, q_2 at the location of q_1 is

$$\vec{B}_{12} = \frac{\mu_0}{4\pi}\frac{q_2}{d^2}\vec{v}_2 \times \hat{d} \qquad \mu_0\text{: vac. permeability,}$$

where $\hat{d}$ is the unit vector along the line between the charges, and the magnetic force on q_1 is

$$\vec{F}_{12}^{\text{mag}} = q_1\vec{v}_1 \times \vec{B}_{12}.$$

17

It follows that

$$F_{12}^{\text{mag}} = \left|\vec{F}_{12}^{\text{mag}}\right| \leqslant \frac{\mu_0}{4\pi}\frac{q_1 q_2}{d^2}v_1 v_2$$

(6) Why " $\leqslant$ ", not " $=$ "? ______________

(7) What's

$$\frac{F_{12}^{\text{mag}}}{F_{12}^{\text{elec}}} \leqslant$$

So, at low speeds $F_{12}^{\text{mag}} \ll F_{12}^{\text{elec}}$ and NTL lives(ish).

18

ADDITIONAL NOTES

§2.4 NSL is the Monarch of Calculations

($\mathbf{8}$) Why?

Since $\vec{\boldsymbol{a}} = \ddot{\vec{\boldsymbol{r}}}$, we obtain from NSL differential equations (equations involving derivatives) that we can attempt to solve.

19

The approach in classical mechanics is to use "force laws" to express what $\vec{\boldsymbol{F}}$ is, then use differential equation ("d.e.") solving techniques to obtain $\vec{\boldsymbol{r}}$ from our knowledge of $\ddot{\vec{\boldsymbol{r}}}$ (through NSL).

Then we can compare the predictions of position with measurements of it (position is the fundamental measurement we make).

20

Examples of force laws are fundamental laws such the universal law of gravitation (also due to our good pal Isaac), Coulomb's law for the electric force between charges, and the law for the magnetic force between moving charges.

There are also approx. laws for frictional forces, etc.

Choosing coordinates wisely is important, when solving the d.e.'s of the problem.

21

§2.5 NSL in Cartesian Coordinates

The velocity, $\vec{\boldsymbol{v}}$, in rectangular xyz coordinates is

$$v_x = \dot{x}, \quad v_y = \dot{y}, \quad v_z = \dot{z}$$

and NSL in these coordinates is

$$F_x = ma_x = m\ddot{x}$$
$$F_y = ma_y = m\ddot{y}$$
$$F_z = ma_z = m\ddot{z}.$$

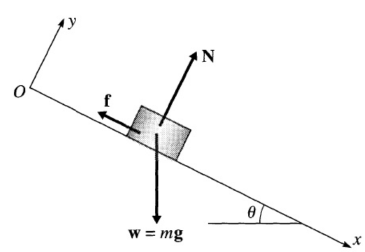
22

Example: A block sliding down an inclined plane.

Choose the x axis to point down the plane and the y axis to be perpendicular to it.

The relevant forces on the block are
1) downward force of gravitation, $m\vec{\boldsymbol{g}}$,
2) normal force of plane on block, $\vec{\boldsymbol{N}} = N\hat{\boldsymbol{y}}$, and
3) friction between plane and block, $\vec{\boldsymbol{f}} = -\mu N\hat{\boldsymbol{x}}$.

23



No force in z direction, so this is a 2-d problem.

24

ADDITIONAL NOTES

Decomposing $m\vec{g}$ into x and y components:

$$mg_x =$$

$$mg_y =$$

There's no net force in the y direction, so all we have to do is work on x. Further,

$$f = |\vec{f}| =$$

25

So, $F_x = m\ddot{x}$ gives us

$$\ddot{x} = \frac{1}{m}$$

Integrating once, then again, we get

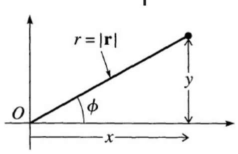
$$v_x = \dot{x} = v_{0_x} + g(sin\,\theta - \mu\, cos\,\theta)t$$

$$x = x_0 + v_{0_x}t + \frac{1}{2}g(sin\,\theta - \mu\, cos\,\theta)t^2$$

26

§2.6 NSL in 2d Polar Coordinates

Many problems, such as orbital motion, are 2-d, and are more tractable in polar coordinates.

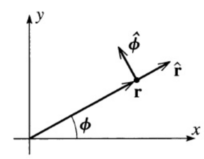


$x =$ $\quad$ $r =$

$y =$ $\quad$ $\phi =$

27

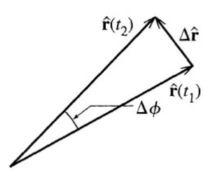
The unit vectors in the r and ϕ directions are shown below:



Their magnitude is fixed, by definition, as ___, but their direction changes from point to point, unlike $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ (fixed in magnitude and direction).

28

Therefore, unlike $\dot{\hat{\boldsymbol{x}}}$ and $\dot{\hat{\boldsymbol{y}}}$, which are both $\vec{\mathbf{0}}$, we need to calculate $\dot{\hat{\boldsymbol{r}}}$ and $\dot{\hat{\boldsymbol{\phi}}}$. Start with $\dot{\hat{\boldsymbol{r}}}$:



The diagram shows $\hat{\boldsymbol{r}}$ at two different times. We see that $\Delta\hat{\boldsymbol{r}}$ points in the direction of $\hat{\boldsymbol{\phi}}$.

29

So

$$\Delta\hat{\boldsymbol{r}} \approx |\hat{\boldsymbol{r}}|\Delta\phi\,\hat{\boldsymbol{\phi}} = \Delta\phi\,\hat{\boldsymbol{\phi}},$$

or

$$\frac{\Delta\hat{\boldsymbol{r}}}{\Delta t} = \frac{\Delta\phi}{\Delta t}\,\hat{\boldsymbol{\phi}}.$$

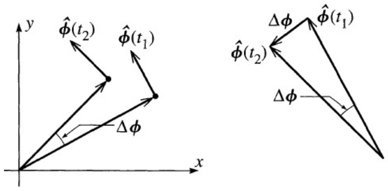
Taking the limit as $\Delta t \to 0$, we get

$$\dot{\hat{\boldsymbol{r}}} = \dot{\phi}\,\hat{\boldsymbol{\phi}}.$$

30

ADDITIONAL NOTES

Next up, $\dot{\hat{\boldsymbol{\phi}}}$:



The diagram shows $\hat{\boldsymbol{\phi}}$ at two different times, and the difference between them when they are parallelly shifted to the same location.

31

This difference points in the $-\vec{\boldsymbol{r}}$ direction, allowing us to conclude that

$$\Delta\hat{\boldsymbol{\phi}} \approx -|\hat{\boldsymbol{\phi}}|\Delta\phi\,\hat{\boldsymbol{r}} = -\Delta\phi\,\hat{\boldsymbol{r}},$$

or

$$\frac{\Delta\hat{\boldsymbol{\phi}}}{\Delta t} = -\frac{\Delta\phi}{\Delta t}\,\hat{\boldsymbol{r}}.$$

Taking the limit as $\Delta t \to 0$, we get

$$\dot{\hat{\boldsymbol{\phi}}} = -\dot{\phi}\,\hat{\boldsymbol{r}}.$$

32

§2.6.1 Velocity in polar coordinates

$$\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}} \qquad \text{and} \qquad \vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}.$$

Therefore,

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}}$$

$$=$$

So,

$$v_r = \dot{r}, \quad \text{and} \quad v_\phi = r\dot{\phi}.$$

33

§2.6.2 Acceleration in polar coordinates

First, take the derivative of $\vec{\boldsymbol{v}}$:

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} =$$

Then substitute from the formulae for $\dot{\hat{\boldsymbol{r}}}$ and $\dot{\hat{\boldsymbol{\phi}}}$:

$$=$$

Finally, regroup as coefficients of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$:

$$\vec{\boldsymbol{a}} = (\qquad\qquad)\hat{\boldsymbol{r}} + (\qquad\qquad)\hat{\boldsymbol{\phi}}$$

34

§2.6.3 Newton's second law (polar coords)

We can decompose the force vector, as well, into radial and angular components:

$$\vec{\boldsymbol{F}} = F_r\hat{\boldsymbol{r}} + F_\phi\hat{\boldsymbol{\phi}}.$$

So, NSL, in these coordinates is

$$F_r =$$

$$F_\phi =$$

35

The quantity $\dot{\phi}$ is called the ______________, usually denoted by ω, and $\ddot{\phi}$ is called the _______ ___________, usually denoted by α.

In terms of these, the components of $\vec{\boldsymbol{v}}$ are

$$v_r = \dot{r}, \quad \text{and} \quad v_\phi = r\omega,$$

and NSL is

$$F_r = ma_r = m(\ddot{r} - r\omega^2)$$

$$F_\phi = ma_\phi = m(r\alpha + 2\dot{r}\omega)$$

36

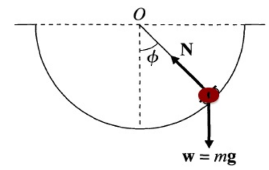
ADDITIONAL NOTES

Compared to the cartesian form of these equations (slide 22), the polar form is more complicated, yet many problems are easier to handle using this form.

37

§2.6.4 Example: motion along an arc of a circle

Situations where this arises are a rock being twirled vertically on a string, a pendulum, or a skateboarder on a circular track.



38 The natural origin is ____________________.

In this problem $r = R$, the fixed radius of the path, so $\dot{r}$ and $\ddot{r}$ are both ___. NSL becomes

$$F_r =$$

$$F_\phi =$$

From the diagram,

$$F_r = mg\cos\phi - N$$

$$F_\phi = -mg\sin\phi$$

39

(9) We are primarily interested in the angular motion in such problems. Equate the two expressions for F_ϕ and get a formula for $\ddot{\phi}$.

40

Since $g > 0$, $R > 0$, we can introduce $k^2 = g/R$ and write the previous equations as

$$\ddot{\phi} = -k^2 \sin\phi.$$

This has no solution in terms of "elementary functions," but we can get a qualitative idea of the behavior it implies. For example:

41

Case 1: $\phi(0) = 0$, $\dot{\phi}(0) = 0$:

Case 2: $\phi(0) = 0$, $\dot{\phi}(0) > 0$:

42

ADDITIONAL NOTES

The last case implies oscillatory behavior. This will occur whenever $\phi \neq 0$ at any value of t (since $\ddot{\phi} = -k^2 \, sin\, \phi$, it will always "push back" on ϕ) and $|\dot{\phi}|$ is not too large.

If the oscillations are small, we'll have $sin\, \phi \approx \phi$ and NSL becomes (approximately)

$$\ddot{\phi} = -k^2 \phi.$$

This can be solved exactly.

43

The general solution is

$$\phi(t) = A\, sin(kt) + B\, cos(kt),$$

where A and B are the two constants of integration that come from the two integrations involved in solving a differential equation with a double derivative.

44

A and B are related to $\phi(0)$ and $\dot{\phi}(0)$.

(10) What's $\phi_0 = \phi(0)$ from the general solution?

$$\phi_0 =$$

(11) What's $\dot{\phi}(t)$ from the general solution?

$$\dot{\phi}(t) =$$

(12) What's $\dot{\phi}_0 = \dot{\phi}(0)$ from this?

$$\dot{\phi}_0 =$$

45

Special case: $\dot{\phi}_0 = 0$.

Then $A = 0$ and the solution is $\phi(t) = \phi_0\, cos(kt)$, clearly oscillatory. The time taken for the object to return to its original position is the period, τ, where $k\tau = 2\pi$, so

$$\tau = \frac{2\pi}{k} = 2\pi \sqrt{\frac{R}{g}}.$$

46

ADDITIONAL NOTES