Arvind Borde / PHY 17, Week 11: Lagrangian Mechanics

§11.1 Hamilton's Principle

§11.1.1 The setup

We use generalized coordinates, $q_i(t)$, to describe the state of a dynamical system at a time t.

Our goal is to determine what the functions $q_i(t)$ are as the system evolves from an initial time t_1 to a final time t_2. We'll refer to this as the (generalized) path that the system takes.

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§11.1.2 Stationary action

To determine the path the system follows, we use a function called the Lagrangian,

$$\mathcal{L} = \mathcal{L}(q_1, q_2, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, t),$$

and the principle that the "action integral"

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\, dt$$

is stationary wrt any one parameter variations of the path $q_i(t)$.

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In other words, compared to nearby paths, the actual path taken by the system is stationary.

That dynamical behavior could be described by this one principle was proposed by William Hamilton, and we call it "Hamilton's Principle," or (somewhat inaccurately) "the principle of least action."

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The stationarity of the action is equivalent to saying that this set of n E-L equations holds (each exactly the same form),

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0.$$

As we'll see, these equations allow us to determine the path the system takes, $q_i(t)$.

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§11.2 What's the Lagrangian?

For a system of N particles, the Lagrangian is

$$\mathcal{L} =$$

where T is the ______ and U the ______.

Why this? Basically, because it works: the E-L equations that follow from Hamilton's principle for this choice of $\mathcal{L}$ are equiv. to NSL in those cases that NSL applies.

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And, when classical particle mechanics is extended to field theory (electromagnetic fields and suchlike), and thence to quantum field theory, Newton's Laws don't hold in their original, literal form.

But, Hamilton's principle works, with a Lagrangian for everything:

$$\mathcal{L}_{\text{all}} = \underbrace{R}_{\text{grav}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{bosons}} + \underbrace{i\bar{\psi}\gamma^{\mu}D_{\mu}\psi}_{\text{fermions}} + \text{Higgs}$$

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ADDITIONAL NOTES

§11.2.1 <u>N unconstrained particles, cartesian coords</u>

We have

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

and, assuming U is conservative,

$$U = U(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2, \ldots \vec{\boldsymbol{r}}_N),$$

where $\vec{\boldsymbol{r}}_i$ is the position of the ith particle, and

$$v_i^2 = \vec{\boldsymbol{v}}_i \cdot \vec{\boldsymbol{v}}_i = \dot{\vec{\boldsymbol{r}}}_i \cdot \dot{\vec{\boldsymbol{r}}}_i.$$

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If we pick our generalized coordinates to be the cartesian components (x_i, y_i, z_i) of each $\vec{\boldsymbol{r}}_i$, we have a system with $n = 3N$ generalized coordinates. Use the notation that

$$q_{i_x} = x_i, \quad q_{i_y} = y_i, \quad q_{i_z} = z_i,$$

we have

$$\frac{\partial \mathcal{L}}{\partial q_{i_x}} = -\frac{\partial U}{\partial x_i} = F_{i_x} \;\&\; \frac{\partial \mathcal{L}}{\partial \dot{q}_{i_x}} = \frac{d(1/2 m_i \dot{x}_i^2)}{d\dot{x}_i} = m_i \dot{x}_i.$$

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So

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{i_x}}\right) = m_i \ddot{x}_i = \dot{p}_{i_x},$$

so we see that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{i_x}}\right) = \frac{\partial \mathcal{L}}{\partial q_{i_x}} \iff \dot{p}_{i_x} = F_{i_x}.$$

In other words, for cartesian coordinates, at least, NSLs for a system of N particles is equivalent to the set of E-L equations.

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As we see above, the role of force is played by

$$\frac{\partial \mathcal{L}}{\partial x_i} \qquad = -\frac{\partial \text{PE}}{\partial x_i}$$

and we call it the "generalized force." The role of momentum is played by

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_{i_x}} \qquad = \frac{\partial \text{KE}}{\partial \dot{q}_{i_x}},$$

and we call it the "generalized momentum."

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Thus, we interpret the E-L equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \quad \text{or} \quad \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

as

$$\text{generalized force} = \dot{\overbrace{\text{generalized momentum}}}$$

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What does "unconstrained" mean above?

(1) What do *you* think it means? ____________

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ADDITIONAL NOTES

§11.2.2 Single particle: 2-d polar coordinates

The generalized coordinates, $\{q_i\}$, here are $\{r, \phi\}$, and the generalized velocities, $\{\dot{q}_i\}$, are $\{\dot{r}, \dot{\phi}\}$.

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}mv^2 - U(r,\phi) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r,\phi),$$

where (of course) m is the mass of the particle.

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(2) What's the r equation?

$$\frac{\partial \mathcal{L}}{\partial r} = \qquad\qquad \frac{\partial \mathcal{L}}{\partial \dot{r}} =$$

So $\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$ becomes

or

$$F_r = -\frac{\partial U}{\partial r} = m(\qquad\qquad)$$

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(3) What's the ϕ equation?

$$\frac{\partial \mathcal{L}}{\partial \phi} = \qquad\qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} =$$

So $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ becomes

$$= \frac{d}{dt}(\qquad\qquad.)$$

Now

$$\vec{\boldsymbol{F}} = F_r\hat{\boldsymbol{r}} + F_\phi\hat{\boldsymbol{\phi}} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}}$$

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Or, $F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}$. So

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi} = rF_\phi.$$

The generalized ϕ-force is just the _______, and the generalized ϕ-momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

16 is the __________________.

So the ϕ E-L equation says

$$\Gamma = \dot{L}.$$

As usual, if

$$\Gamma = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

then

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

is conserved. This is true in general.

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§11.2.3 Generalized conservation laws

The E-L equation written as

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

immediately tells us that if $\mathcal{L}$ is independent of a particular coordinate, q_i, then (since $\partial\mathcal{L}/\partial q_i = 0$) the corresponding generalized momentum $\partial\mathcal{L}/\partial\dot{q}_i$ is conserved.

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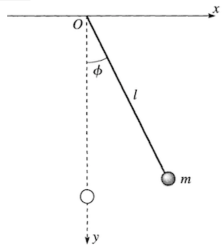
ADDITIONAL NOTES

§11.3 Systems With Constraints

§11.3.1 Example: a single pendulum

The mass m moves in the x-y plane, but the coordinates are *constrained* by

$$x^2 + y^2 = \ell^2$$



An obvious generalized coordinate here is ______ ______. Clearly this single one suffices.

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With the origin at the point of suspension,

$$x =$$

$$y =$$

If the zero of PE is at the bottommost position (equilibrium position), then

$U =$

and

20 $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

(4) What's the Lagrangian?

$$\mathcal{L} =$$

(5) What are these?

$$\frac{\partial \mathcal{L}}{\partial \phi} = \qquad\qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} =$$

(6) Write down the E-L eqn. here:

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The torque exerted by gravity on m is

$$\Gamma = (-mg)(\ell)\, sin\, \theta,$$

and the moment of inertia of m is $I = m\ell^2$. So the E-L equation is the familiar result that

$$\Gamma = \dot{L} = (\dot{I\omega}) = I\alpha,$$

where $\omega = \dot{\phi}$ is the angular velocity, and $\alpha = \ddot{\phi}$ is the angular acceleration.

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§11.3.2 Degrees of Freedom

This is the number of coordinates that can be independently varied. For example, although a single pendulum moves in the 2-d x-y plane, it has only one degree of freedom because varying either x or y determines how the other changes.

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An N-particle system in 3-d has ____ coordinates.

If it also has $3N$ degrees of freedom, we say the system is unconstrained.

If it has fewer than $3N$ degrees of freedom, we say the system is constrained.

If a system with n degrees of freedom needs n generalized coordinates, it is called __________.

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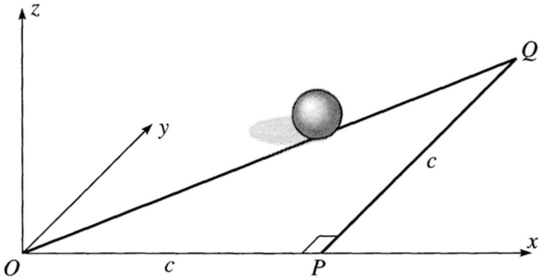
ADDITIONAL NOTES

§11.3.3 A nonholonomic system: an example



Although the position of the ball requires just two coordinates, x and y, making this a system with 2 degrees of freedom, its rotational state requires 3 more variables, and so a total of 5 generalized coordinates.

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§11.3.4 Holonomic systems

We'll confine ourselves to holonomic systems.

We've shown that for unconstrained systems, the E-L equations are equivalent to NSL.

We'll now examine the status of the E-L equations for constrained systems.

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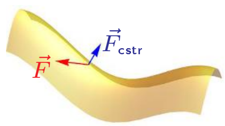
Let's look at a particle constrained to move on a 2-d surface in 3-d space, as an example.

There will be two types of forces on the particle, $\vec{\boldsymbol{F}}_{\text{cstr}}$, the net force that constrains the particle *to* the surface, and $\vec{\boldsymbol{F}}$, the net force that causes motion *along* the surface.



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$\vec{\boldsymbol{F}}_{\text{cstr}}$ may or may not be conservative, but we assume that $\vec{\boldsymbol{F}}$ obeys the second condition for conservatism (path-independence of the work integral), so there's a PE, $U(\vec{\boldsymbol{r}}, t)$, such that

The total force is

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Choose the Lagrangian here, too, as

$$\mathcal{L} =$$

(7) Does anything seem excluded here? ____

____

But, as we'll see, it comes out OK in the wash, as long as we only consider paths constrained to lie on the surface of interest.

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Let $\vec{\boldsymbol{r}}(t)$ be the path the particle follows, and

$$\vec{\boldsymbol{R}}(t) = \vec{\boldsymbol{r}}(t) + \vec{\boldsymbol{\epsilon}}(t)$$

be an infinitesimally nearby path *on the surface*.

We assume that $\vec{\boldsymbol{R}}(t)$ has the same endpoints as $\vec{\boldsymbol{r}}(t)$, so $\vec{\boldsymbol{\epsilon}}(t_1) = \vec{\boldsymbol{\epsilon}}(t_1) =$ ___.

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ADDITIONAL NOTES

Now

$\mathcal{L}(\vec{\boldsymbol{R}}, \dot{\vec{\boldsymbol{R}}}, t)$

$=$

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The difference in the Lagrangian between the paths $\vec{\boldsymbol{R}}(t)$ and $\vec{\boldsymbol{r}}(t)$ is

$$\delta\mathcal{L} =$$

Therefore, the difference in the action integral between the paths $\vec{\boldsymbol{R}}(t)$ and $\vec{\boldsymbol{r}}(t)$ is

$$\delta S = \int_{t_1}^{t_2} \delta\mathcal{L}\, dt = \int_{t_1}^{t_2} (m\dot{\vec{\boldsymbol{r}}} \cdot \dot{\vec{\boldsymbol{\epsilon}}} - \vec{\boldsymbol{\epsilon}} \cdot \vec{\nabla} U)\, dt$$

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But, using integration by parts,

$$\int_{t_1}^{t_2} (m\dot{\vec{\boldsymbol{r}}} \cdot \dot{\vec{\boldsymbol{\epsilon}}})\, dt =$$

So

$$\delta S = -\int_{t_1}^{t_2} \vec{\boldsymbol{\epsilon}} \cdot (m\ddot{\vec{\boldsymbol{r}}} + \vec{\nabla} U)\, dt$$

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Now NSL says that

$$m\ddot{\vec{\boldsymbol{r}}} =$$

Meanwhile

$$\vec{\nabla} U =$$

So

$\delta S =$

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But $\vec{\boldsymbol{\epsilon}}$ is a vector from the path $\vec{\boldsymbol{r}}(t)$ to the path $\vec{\boldsymbol{R}}(t)$, and both paths are on the surface of constraint, i.e., $\vec{\boldsymbol{\epsilon}}$ is tangent to the surface.

But $\vec{\boldsymbol{F}}_{\text{cstr}}$ is normal to the surface. So $\vec{\boldsymbol{\epsilon}} \cdot \vec{\boldsymbol{F}}_{\text{cstr}} = \underline{\quad}$. Therefore $\delta S = 0$ and is therefore stationary wrt variations *that obey the constraints*.

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If we choose two generalized coordinates, q_1 and q_2, on the surface it follows from the stationarity of S on the surface that the E-L equation will hold for our choice of $\mathcal{L}$,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

This is true in general dimensions.

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ADDITIONAL NOTES