Arvind Borde / MTH 675, Unit 9: Topology and Manifolds

1. Introduction

The basic idea of an nd manifold, $\mathcal{M}$, ____

We are trying to capture the notion of an entity that locally "looks like" $\mathbb{R}^n$ – but this naive idea needs considerable amplification.

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2. Topological Spaces

The first requirement on $\mathcal{M}$ is that it be a topological space.

Intuitively, this is a set to which the notion of 'closeness' can be carried over from the domain of real numbers, without also carrying along the other properties of order, distance, etc., with which the real number system is endowed.

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Definition [Topology]:
A topology on a set S is a collection $\mathcal{T}$ of subsets of S with these properties:

- ________________________________
- ________________________________
- ____

3

Definition [Topological space] If $\mathcal{T}$ is a topology on a set S, the pair $(S, \mathcal{T})$ is called a topological space.

Definition [Open set] Let $(S, \mathcal{T})$ be a topological space. A subset O of S is called open if belongs to the collection $\mathcal{T}$.

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The axioms for a topological space are customarily stated in topology textbooks as if they were instructions from heaven, but they represent the distillation of many years of discussion. The aim of this discussion was to capture within a single set of axioms as large a variety of systems as possible to which the notion of closeness (and the related notion of continuity) applies.

The point of this axiomatization, and of the related definitions given above, is that an open set is meant to represent a generalization of the notion of the 'neighborhood' of a point (i.e., of the set of points around a given point).

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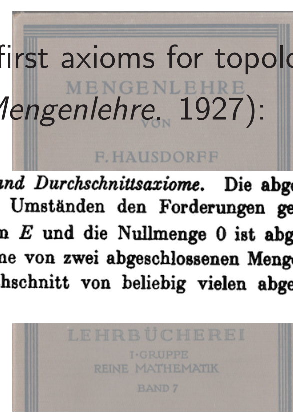
Possibly, the first axioms for topology (Hausdorff, *Mengenlehre*, 1927):



I. ***Summen- und Durchschnittsaxiome.*** **Die abgeschlossenen Mengen sollen unter allen Umständen den Forderungen genügen:**

(1) Der Raum E und die Nullmenge 0 ist abgeschlossen.

(2) Die Summe von zwei abgeschlossenen Mengen ist abgeschlossen.

(3) Der Durchschnitt von beliebig vielen abgeschlossenen Mengen ist abgeschlossen.

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ADDITIONAL NOTES

For a concrete example of a topological space (in fact, as stated, the inspiration for the general notion), let S be the set of real numbers, $\mathbb{R}$, and let $\mathcal{T}$ be the collection of all possible unions of all the open intervals of $\mathbb{R}$ (including the empty interval and $\mathbb{R}$ itself). Then $\mathcal{T}$ is a topology on $\mathbb{R}$.

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(1) Sticking with that example, why might we need the restriction to finite intersections?

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Definition [Closed set] Let $(S, \mathcal{T})$ be a topological space. A subset C of S is called closed if ______ ______________.

It follows from this definition that arbitrary intersections and finite unions of closed sets are closed. (2) What's an example of an infinite union of closed sets that's not closed?

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Definition [Interior] Let $(S, \mathcal{T})$ be a topological space and A a subset S. The interior of A is ____ (i.e, the largest open set contained in A).

10

Definition [Closure] Let $(S, \mathcal{T})$ be a topological space and let A be a subset of S. The closure of A, denoted by $\bar{A}$, is ______________________ ____________________________ (i.e., the smallest closed set containing A).

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Theorem Let $(S, \mathcal{T})$ be a topological space, let A be a subset of S, and let $x \in S$. Then $x \in \bar{A}$ if and only if every open set O containing x intersects A.

The proof of this and all subsequent topological theorems may be found in any topology text.

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ADDITIONAL NOTES

The closure of a set A represents, roughly speaking, the set of points either in A or 'infinitely close' to it.

And, still speaking very roughly, the previous theorem asserts that a point belongs to $\bar{A}$ if and only if every neighborhood of x intersects A.

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Definition [Limit point] Let $(S, \mathcal{T})$ be a topological space and let A be a subset S. A point $x \in S$ is called a limit point of A if ____________

____

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Theorem Let $(S, \mathcal{T})$ be a topological space and let A be a subset S. Then A is closed if and only if it contains all its limit points.

Proof: One first shows that

$$\bar{A} = A \cup \{\text{all the limit points of } A\}.$$

The result then follows from the fact that A is closed if and only if $A = \bar{A}$.

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Definition [Boundary] Let $(S, \mathcal{T})$ be a topological space and let A be a subset S. The boundary of A, denoted by $\dot{A}$, is defined to be ___________

(3) What does a dot on A suggest to you? _____

____

Is there a connection?

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Consider a sphere of radius r.

(4) What's its volume? ____________

(5) What's its boundary? ________________

(6) What's the surface area? ____________

(7) Is there a derivative connection? _____

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Definition [Hausdorff space] A topological space $(S, \mathcal{T})$ is called Hausdorff if for every pair x and y of distinct points in S __________________

____.

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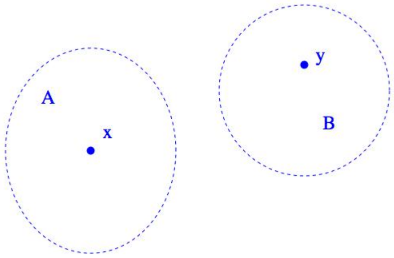
ADDITIONAL NOTES

This is sometimes called a *separation property* for it asserts (roughly) that distinct points can be enclosed in non-intersecting neighborhoods.



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An example of a non-Hausdorff space may be constructed by taking two real lines, each with the standard topology (defined above) and identifying the regions $x < 0$ on the two. The resulting space is still a topological space, but the Hausdorff property fails to hold at the two origins (still distinct under the stated identification).

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Definition [Separation] Let $(S, \mathcal{T})$ be a topological space. A pair of disjoint open sets O_1 and O_2 is called a separation of S if $O_1 \cup O_2 = S$.

Definition [Connected space] A topological space $(S, \mathcal{T})$ is called connected if there does not exist a separation of S.

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Definition [Covering] Let S be a set and let $\mathcal{B}$ be a collection of subsets of S. Then $\mathcal{B}$ is called a covering of S if the union of the elements of $\mathcal{B}$ equals S.

Definition [Open covering] Let $(S, \mathcal{T})$ be a topological space. A covering $\mathcal{B}$ of S is called an open covering if its elements are open sets.

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Definition [Compact space] A topological space $(S, \mathcal{T})$ is called compact if ____ (i.e., a finite subcollection of $\mathcal{B}$ that still covers S).

This isn't the most transparent of definitions, but it tries to capture the notion of a space that is 'finite' in extent.

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Theorem A closed subset of a compact space is compact.

Theorem A compact subset of a Hausdorff space is closed.

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ADDITIONAL NOTES

Definition [Product topology] Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be two topological spaces. The product topology on $S_1 \times S_2$ is the collection of all unions of all sets of the type $O_1 \times O_2$, where O_1 and O_2 are open sets of $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$, respectively.

Theorem A finite product of compact spaces is compact.

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A set with a topology defined on it induces a topology on its subsets in a natural manner.

Definition [Subspace topology] Let $(S, \mathcal{T})$ be a topological space and let P be a subset of S. Then the collection

$$\mathcal{T}_P = \{P \cap O \mid O \in \mathcal{T}\}$$

is a topology on P, called the subspace topology.

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Theorem The product of two Hausdorff spaces is Hausdorff. A subspace of a Hausdorff space is Hausdorff.

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3. Homeomorphisms

A homeomorphism between two topological spaces is a statement of abstract identity: homeomorphic spaces are the same space as far as topological properties go.

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Definition [Continuous function] Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be two topological spaces. A function $f : S_1 \to S_2$ is called continuous if for each open subset O of S_2, the inverse image $f^{-1}(O)$ is an open subset of S_1.

Seems unintuitive, at first, but this tries to capture the idea that nearby points in S_2 come from nearby points in S_1.

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The preceding definition is equivalent to this:

First define f to be continuous at a point $x \in S_1$ if for every open set $O_2 \subset S_2$ there is an open $O_1 \subset S_1$ such that $f(O_1) \subset O_2$.

Then, define f to be continuous if it is continuous at every point.

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ADDITIONAL NOTES

Definition [Homeomorphism] Topological spaces $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ are called homeomorphic if there is a one-to-one mapping f between them such that f and f^{-1} are both continuous.

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4. Atlases

Definition [Chart] An n-dimensional chart on a Hausdorff topological space, $\mathcal{M}$, is a

____

_________________________.

The existence of a chart means that the set O has the topological properties of an open subset of $\mathbb{R}^n$ and *inherits its dimensionality*.

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Definition [Atlas] An n-dimensional smooth atlas of a Hausdorff topological space, $\mathcal{M}$, is a collection of open sets, O_i, each with a chart, σ_i, such that

- , and
- ____
  __________________________________.

The second says that charts are smooth functions of each other in overlap regions.

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5. Manifolds

Definition [Manifold] An n-dimensional manifold, $\mathcal{M}$, is a Hausdorff topological space equipped with an n-dimensional smooth atlas.

This is a barebones entity, so far, In order to be able to talk about curvature, etc., we'll need additional structure.

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ADDITIONAL NOTES