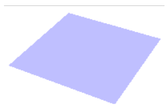
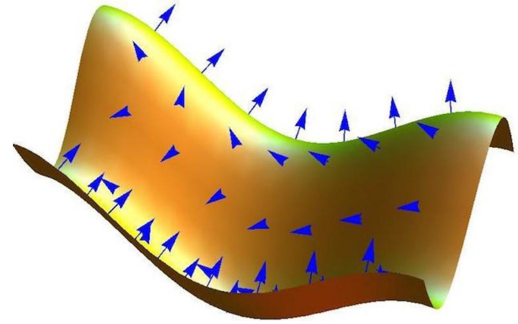
1. Surfaces (2d) in $\mathbb{R}^3$

We'll assume, for now, that we know intuitively what a (two-dimensional) surface looks like in $\mathbb{R}^3$,



1 and also have some intuitive notion of curvature.

For a smooth surface, we expect there to be a normal (perpendicular) at every point:



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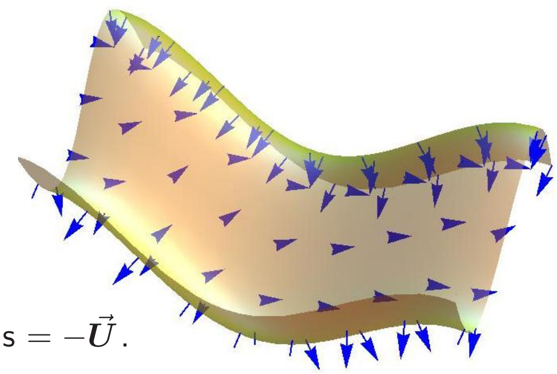
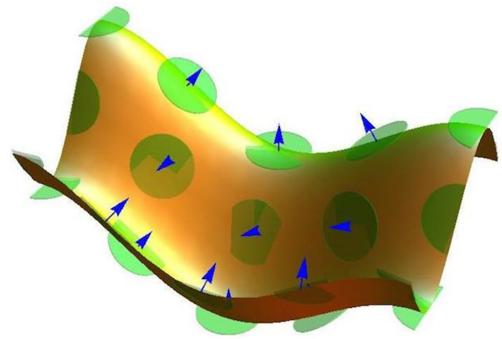
Actually, two (every normal has an evil twin, its negative):



If $\vec{U}$ is a normal, so is $= -\vec{U}$.

3

There are also (unit) tangent vectors at each point, spanning a tangent plane.



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Pick a point **P** on a surface, $\mathcal{M} \subset \mathbb{R}^3$, and let $\vec{U}$ be a unit normal at that point.

Picking one of the unit tangent vectors, $\vec{V}$, at **P**, the plane spanned by $\vec{U}$ and $\vec{V}$ will intersect the surface at a curve, $\alpha_{\mathbf{V}}$.

We call $\alpha_{\mathbf{V}}$ the ____________________ of $\mathcal{M}$ in the direction $\vec{V}$.

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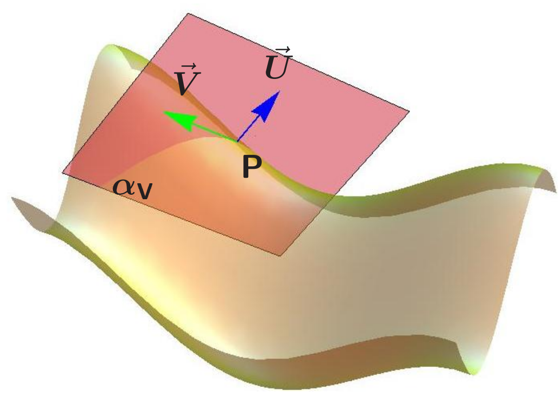

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ADDITIONAL NOTES

(1) Had we picked $-\vec{U}$ as the normal, would we get the same $\boldsymbol{\alpha}_{\mathbf{V}}$ or an "opposite" one?

____

So, how does the choice of normal to a surface affect our discussion?

Geometrically one of $\vec{U}$, $-\vec{U}$ will be the same as something we've seen, the other will be its oppo.

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(2) But what's this "something" we've seen?____

In our one-dimensional discussion, that vector was *defined* as ___′, hence fully determined.

Here, the apparent ambiguity in the normal to a surface, $\vec{U}$, means that we have $\vec{U} =$ ______.

Keep this in mind, and get back to the curve $\boldsymbol{\alpha}_{\mathbf{V}}$.

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$\boldsymbol{\alpha}_{\mathbf{V}}$ will have some curvature at the point **P**, $k_{\mathbf{P}}$.

There will be a normal to the surface $\mathcal{M}$ at **P**, $\vec{U}$.

We define the ________________________________ ________________________, $k_{\mathbf{U}}(\vec{V})$, as

$$k_{\mathbf{U}}(\vec{V}) \equiv \begin{cases} \quad , & \text{if } \vec{U} = \vec{N} \\ \quad , & \text{if } \vec{U} = -\vec{N} \end{cases}$$

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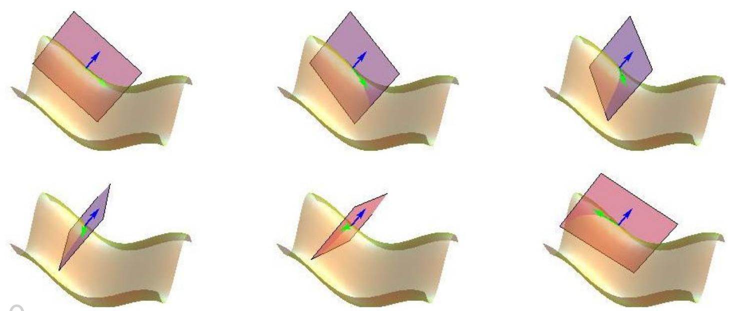
For fixed $\vec{U}$, different normal sections at **P** have different curvatures, $k_{\mathbf{U}}(\vec{V})$, depending on $\vec{V}$:



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There are two directions, $\vec{V}_1$ and $\vec{V}_2$, where $k_{\mathbf{U}}(\vec{V})$ has minimum and maximum values, k_1 and k_2, called the ____________________________ at **P**.

Further $\vec{V}_1$ and $\vec{V}_2$ are orthogonal.

The __________________________, K, of $\mathcal{M}$ at **P** is defined as

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The existence of maximum and minimum values for k is an extension of Rolle's theorem.

As $\vec{V}$ rotates through 2π in the tangent plane, k returns to its starting value, *as does its derivative*.

(Showing the directions are orthogonal requires further argument.)

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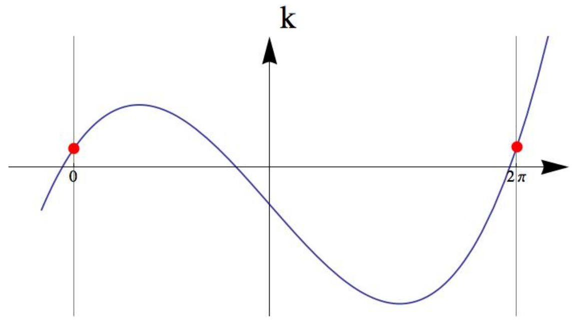
ADDITIONAL NOTES



$k(0) = k(2\pi) \qquad k'(0) = k'(2\pi)$

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Had we picked $-\vec{U}$ as the unit normal what would happen to

(3) k_1? ________

(4) k_2? ________

(5) K? ________

Does K capture our intuitive ideas of curvature?

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To support intuition, we need machinery.

In 3d if a surface is represented by $z = f_z(x, y)$, then the *set* of level surfaces will be

$$z = f_z(x, y) + w,$$

or

$$w = z - f_z(x, y).$$

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The normal is the gradient of w,

$$\nabla w = \nabla\big(z - f_z(x, y)\big)$$

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2. Examples

Example 1: A plane,
Let's pick $\mathcal{M}$ to be the plane $z =$ constant.

With the usual Euclidean norm
(6) What are the unit normals? $\vec{U} =$ ________
(7) What are the unit tangents?
Of the form $\vec{V} =$ (____, ____, __),
where θ is the angle $\vec{V}$ makes with the x-axis.

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The plane spanned by $\vec{U}$ and any of these tangents, will intersect $\mathcal{M}$ in a ________.

Thus, for any $\vec{V}$, $\alpha_{\mathbf{V}}$ has ____ curvature.

So

$$K = k_1 k_2 =$$

as expected.

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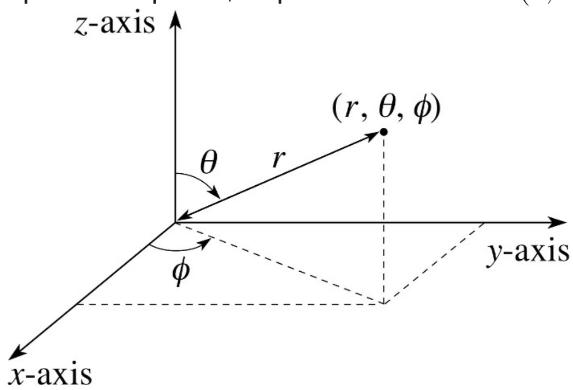
ADDITIONAL NOTES

Example 2: A sphere, in polar coordinates (r, θ, ϕ).



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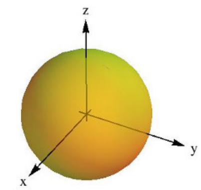
Let's pick $\mathcal{M}$ to be the sphere $r = R$.

(8) What are the unit normals? $\vec{U} =$ ______

The unit tangents at an arbitrary point are hard to calculate, but pick **P** to be the point $(R, 0, 0)$.

(9) Plot **P**, $\vec{U}$.



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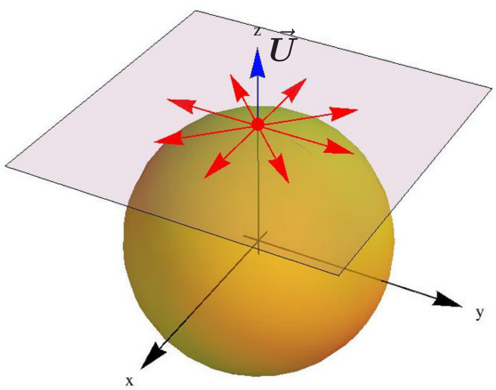
Tangents & tangent plane



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Clearly, for any other choice of point, **P**, the normal sections will again be great circles.

(10) What's the curvature, k, of any of these great circles? __________

(11) What's the Gauss curvature, K of a sphere?

____

(12) Reasonable? __________

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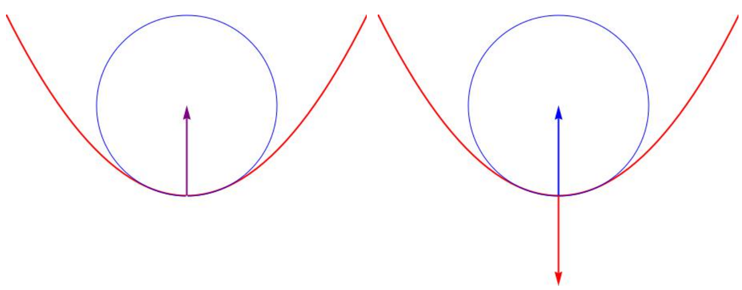
I was slimy about something up there. Hint:



(13) What?

23

____

____

____

So, if the normals to a sphere point outward, the individual principal curvatures are both ______

Despite that fudgery, K_{sphere} is still $1/R^2_{\text{sphere}}$.

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ADDITIONAL NOTES

Example 3: A saddle (hyperbolic paraboloid)
Defined by the formula

$$z = a(y^2 - x^2)$$

(14) What is its cross-section in the y-z plane?

____

(15) What is its cross-section in the x-z plane?

____

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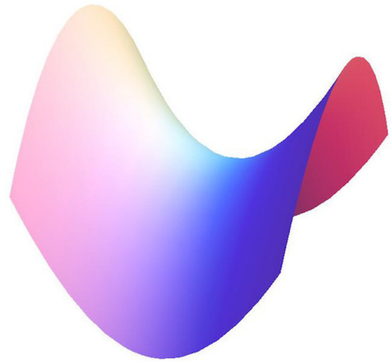
What's a "saddle"?



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If $w = z - ay^2 + ax^2$, then
(16) what's ∇w?

$$\nabla w = (\partial_x w, \partial_y w, \partial_z w)$$
$$= (\_\_\_\_, \_\_\_\_\_, \_\_)$$

(17) What's the unit normal at the origin? ______

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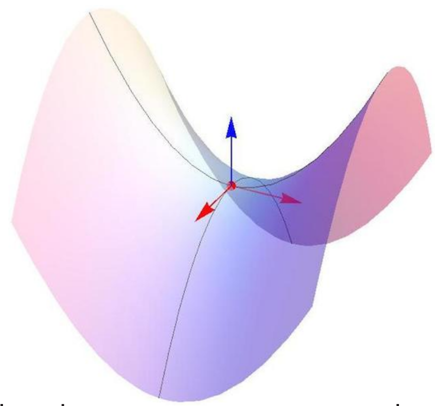

A normal and two tangents to normal sections.

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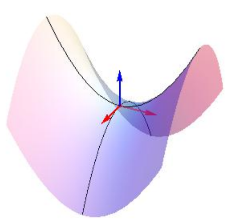
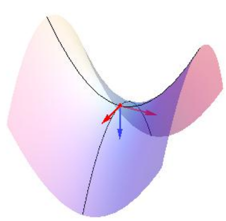
Now, no matter whether we pick





we get one principal curvature ___0, and one ___0.

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(18) So, at the origin, K_{saddle}____

With a suitable definition of a "saddle" we can show that this is true everywhere.

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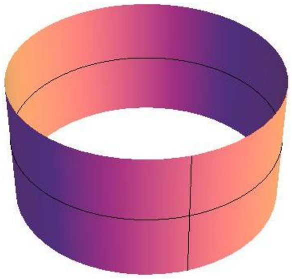
ADDITIONAL NOTES

Example 4: A cylinder, radius R.



The principal normal sections are shown.

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(19) For the circular principal section, $k =$ ______.

(20) For the "vertical" section, $k =$ ___.

(21) Therefore, $K_{\text{cylinder}} =$ ___

Shocker?

Intrinsic ("local") geometry *is* flat.

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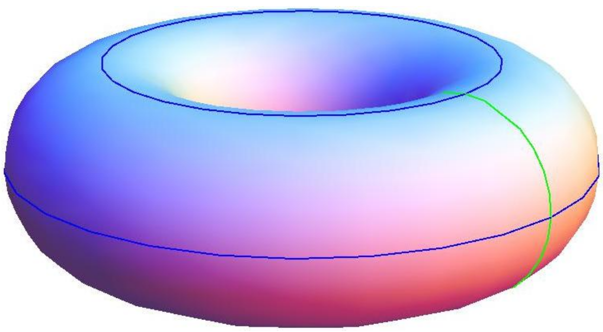
Example 5: A torus.



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(22) What are the principal curvatures outside, on the "top," and inside?

____

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ADDITIONAL NOTES