Arvind Borde / MTH 675, Unit 7: Curves and Curvature V

0. Reminders

R1) A derivative mapping, F_*, of a mapping, F, maps vectors to vectors.

The vector $\vec{V}_{\mathbf{P}}$ gets mapped to $F_*(\vec{V}_{\mathbf{P}})$ by this procedure: $\boldsymbol{\alpha}$ is the straight line through **P** in the direction of $\vec{V}_{\mathbf{P}}$; $\boldsymbol{\beta}$ is its image under F. Then

$$F_*(\vec{V}_{\mathbf{P}}) = \boldsymbol{\beta}'(0).$$

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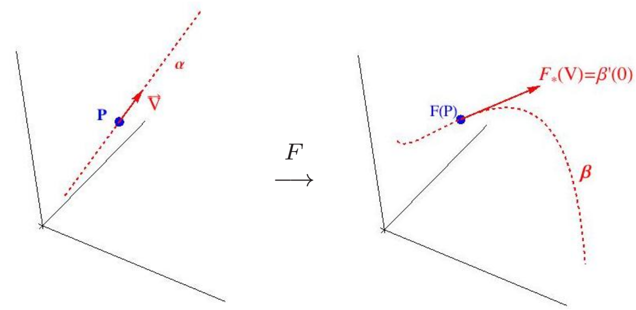
Pictorially:



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R2) An isometry is a distance preserving mapping:

$$d\big(F(\mathbf{P}), F(\mathbf{Q})\big) = d(\mathbf{P}, \mathbf{Q})$$

where, as usual, each point in $\mathbb{R}^n$ can be identified with a vector.

Special isometries:
Translations (T), orthogonal transformations (C).

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T: Shift by a fixed vector, say $\vec{A}$.

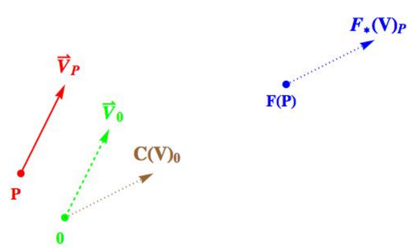
Cs are inspired by rotations:
_____ transformations, preserving ___ products.
(**1**) Is there an ortho that's not a rotation?
____

4 An isometry F can be written as $F =$ _____

1. Isometries and Derivative Maps (D-Maps)

For any isometry F, with orthogonal part C,

$$F_*(\vec{V}_{\mathbf{P}}) = \big(C(\vec{V})\big)_{F(\mathbf{P})}$$



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OK, why?

From the definition, $F_*(\vec{V}_{\mathbf{P}})$ is the initial tangent vector to the curve $F(\mathbf{P} + t\vec{V})$. (Strictly, that's the image of the curve.) But

$$F(\mathbf{P} + t\vec{V}) = TC(\mathbf{P} + t\vec{V}) =$$

$$=$$

where $\vec{A}$ is the "shift vector" of T.

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But, $F(\mathbf{P}) = T\big(C(\mathbf{P})\big) =$.

Therefore,

$$F(\mathbf{P} + t\vec{V}) =$$

We'd said $F_*(\vec{V}_\mathbf{P})$ is the initial tangent vector to the curve $F(\mathbf{P} + t\vec{V})$. From above that's clearly

$$\big(C(\vec{V})\big)_{F(\mathbf{P})}$$

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D-Maps of isometries preserve dot products:

$$\vec{V}_\mathbf{P} \cdot \vec{W}_\mathbf{P} =$$

Why? Because an orthogonal transformation does. (______________________)

$$F_*(\vec{V}_\mathbf{P}) \cdot F_*(\vec{W}_\mathbf{P}) =$$
$$=$$

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Now, playing "fast and loose" again (and invoking the goddesses of isomorphism) it's easy to see that dot products are the same, no matter what the "location" of the vector.

Thus

$$(\vec{V})_{F(\mathbf{P})} \cdot (\vec{W})_{F(\mathbf{P})} =$$

and we're done.

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2. Isometries and Frames

Let $\{\vec{E}_1, \vec{E}_2, \vec{E}_3\}$ be an orthonormal frame.

(2) What are your finely tuned instincts about $\{F_*(\vec{E}_1), F_*(\vec{E}_2), F_*(\vec{E}_3)\}$?

____

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Now that we're talkin' two orthonormal frames, let's really talk two orthonormal frames:

(3) How might they be related?

____

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Is there an isometry at all?

1) Let C be the unique linear transformation that sends the coordinates of $\vec{E}$ to those of $\vec{F}$. Such a transformation preserves dot products, and is therefore orthogonal.

2) Let T be translation by $\mathbf{Q} - C(\mathbf{P})$.

3) Let $F = TC$, and let's get to business:

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(4) What's $F(\mathbf{P})$?

$$F(\mathbf{P}) = T\big(C(\mathbf{P})\big) =$$

Then

$$F_*(\vec{\boldsymbol{E}}_i) =$$

And, since C is unique, so is F.

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3. Isometries and Orientation

If the orientations of the two frames $\{\vec{\boldsymbol{E}}_1, \vec{\boldsymbol{E}}_2, \vec{\boldsymbol{E}}_3\}$ and $\{\vec{\boldsymbol{F}}_1, \vec{\boldsymbol{F}}_2, \vec{\boldsymbol{F}}_3\}$ are the same (both $+1$ or -1), F is called ______________________________.

If they are different, F is called ______________ ______________.

Define: $\text{sgn}F \begin{cases} +1 & \text{when } F \text{ preserves orientation.} \\ -1 & \text{when } F \text{ reverses orientation.} \end{cases}$

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Example of an orientation reversing isometry: ______________

It follows (from components) that

$$F_*(\vec{\boldsymbol{V}} \times \vec{\boldsymbol{W}}) = \text{sgn}F(F_*(\vec{\boldsymbol{V}}) \times F_*(\vec{\boldsymbol{W}}))$$

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4. Isometries and Curves

Let $\boldsymbol{\alpha}$ be a curve with unit tangent $\vec{\boldsymbol{T}}$, curvature k and torsion τ. Let $\hat{\boldsymbol{\alpha}} = F(\boldsymbol{\alpha})$ be its image under an isometry F. Then

$$\hat{k} =$$

$$\hat{\tau} =$$

$$\hat{\vec{\boldsymbol{T}}} =$$

$$\hat{\vec{\boldsymbol{N}}} =$$

$$\hat{\vec{\boldsymbol{B}}} =$$

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In order to show this, one first shows that that isometries preserve "velocities" and "accelerations".

The velocity of a curve, $\boldsymbol{\alpha}(t)$, is simply its first derivative (aka its tangent):

$$\boldsymbol{\alpha}'(t) =$$

The acceleration is the second derivative:

$$\boldsymbol{\alpha}''(t) =$$

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If $\boldsymbol{\alpha}$ is a curve, and $\vec{\boldsymbol{V}}$ is "vector field" defined along it, then $\hat{\vec{\boldsymbol{V}}} = F_*(\vec{\boldsymbol{V}})$ will be a vector field along the image curve $\hat{\boldsymbol{\alpha}} = F(\boldsymbol{\alpha})$.

Then by examining components, it follows that

$$\hat{\vec{\boldsymbol{V}}}' = F_*(\vec{\boldsymbol{V}}')$$

With this in hand, we can proceed.

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Step 1: The tangent and the curvature.

$$\|\hat{\boldsymbol{\alpha}}'(t)\| = \|F_*(\boldsymbol{\alpha}')\| = \|\boldsymbol{\alpha}'\| = 1.$$

Therefore

$$\hat{\vec{\boldsymbol{T}}} = \hat{\boldsymbol{\alpha}}' = F_*(\boldsymbol{\alpha}') = F_*(\vec{\boldsymbol{T}})$$

And

$$\hat{k} = \|\hat{\vec{\boldsymbol{T}}}'\| = \|\hat{\boldsymbol{\alpha}}''\| = \|F_*(\boldsymbol{\alpha}'')\| = \|\boldsymbol{\alpha}''\| = k$$

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Step 2: The normal.

Remember that $\vec{\boldsymbol{N}} = \vec{\boldsymbol{T}}'/k = \boldsymbol{\alpha}''/k$. Then

$$\hat{\vec{\boldsymbol{N}}} = \frac{\hat{\boldsymbol{\alpha}}''}{\hat{k}} = \frac{F_*(\boldsymbol{\alpha}'')}{k} = F_*\left(\frac{\boldsymbol{\alpha}''}{k}\right) = F_*(\vec{\boldsymbol{N}})$$

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Step 3: The binormal.

$$\begin{aligned}\hat{\vec{\boldsymbol{B}}} &= \hat{\vec{\boldsymbol{T}}} \times \hat{\vec{\boldsymbol{N}}} \\ &= F_*(\vec{\boldsymbol{T}}) \times F_*(\vec{\boldsymbol{N}}) \\ &= (\text{sgn}F)F_*(\vec{\boldsymbol{T}} \times \vec{\boldsymbol{N}}) \\ &= (\text{sgn}F)F_*(\vec{\boldsymbol{B}}).\end{aligned}$$

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Step 4: The torsion.

We had defined this via $\vec{\boldsymbol{B}}' = -\tau\vec{\boldsymbol{N}}$.

Therefore, $\tau = -\vec{\boldsymbol{B}}' \cdot \vec{\boldsymbol{N}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{N}}'$, and

$$\begin{aligned}\hat{\tau} &= \hat{\vec{\boldsymbol{B}}} \cdot \hat{\vec{\boldsymbol{N}}}' \\ &= (\text{sgn}F)F_*(\vec{\boldsymbol{B}}) \cdot F_*(\vec{\boldsymbol{N}}') \\ &= (\text{sgn}F)\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{N}}' \\ &= (\text{sgn}F)\tau\end{aligned}$$

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5. Congruence of Curves

We're now in a position to show that if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are curves with unit tangents, such that $k_\alpha = k_\beta$ and $\tau_\alpha = \pm\tau_\beta$, then they're congruent.

A) Fix a parameter value, say 0, for $\boldsymbol{\alpha}$. Let

$$\{\vec{\boldsymbol{T}}_\alpha(0), \vec{\boldsymbol{N}}_\alpha(0), \vec{\boldsymbol{B}}_\alpha(0)\}$$

be the "Frenet frame" at $\boldsymbol{\alpha}(0)$.

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B) Let

$$\{\vec{\boldsymbol{T}}_\beta(0), \vec{\boldsymbol{N}}_\beta(0), \vec{\boldsymbol{B}}_\beta(0)\}$$

be the Frenet frame at $\boldsymbol{\beta}(0)$.

Let F be the isometry that carries

$\{\vec{\boldsymbol{T}}_\alpha(0), \vec{\boldsymbol{N}}_\alpha(0), \vec{\boldsymbol{B}}_\alpha(0)\}$

to $\{\vec{\boldsymbol{T}}_\beta(0), \vec{\boldsymbol{N}}_\beta(0), \pm\vec{\boldsymbol{B}}_\beta(0)\}$.

where we use the plus sign if $\tau_\alpha = \tau_\beta$ and the minus if $\tau_\alpha = -\tau_\beta$.

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C) Let $\hat{\boldsymbol{\alpha}} = F(\boldsymbol{\alpha})$. Then

$$\hat{\boldsymbol{\alpha}}(0) = \boldsymbol{\beta}(0) \qquad \hat{\vec{\boldsymbol{T}}}(0) = \vec{\boldsymbol{T}}(0)$$
$$\hat{k} = k_\beta \qquad \hat{\vec{\boldsymbol{N}}}(0) = \vec{\boldsymbol{N}}(0)$$
$$\hat{\tau} = \tau_\beta \qquad \hat{\vec{\boldsymbol{B}}}(0) = \vec{\boldsymbol{B}}(0)$$

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D) The next step involves showing that $\boldsymbol{\beta}$ is a translation of $\hat{\boldsymbol{\alpha}} = F(\boldsymbol{\alpha})$.

But, since $\hat{\boldsymbol{\alpha}}(0) = \boldsymbol{\beta}(0)$, it follows that $\boldsymbol{\beta} = F(\boldsymbol{\alpha})$.

We do this by defining

$$f = \hat{\vec{\boldsymbol{T}}} \cdot \vec{\boldsymbol{T}} + \hat{\vec{\boldsymbol{N}}} \cdot \vec{\boldsymbol{N}} + \hat{\vec{\boldsymbol{B}}} \cdot \vec{\boldsymbol{B}}$$

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By taking its derivative, we can show that $f' = 0$. But we know that $f(0)$ is 3. Therefore $f = 3$ throughout.

Now, each of the dot products in f is less than or equal to 1 from the Schwarz inequality (they are unit vectors), with equality holding if and only if the vectors are equal:

$$\hat{\vec{\boldsymbol{T}}} = \vec{\boldsymbol{T}}, \quad \hat{\vec{\boldsymbol{N}}} = \vec{\boldsymbol{N}}, \quad \hat{\vec{\boldsymbol{B}}} = \vec{\boldsymbol{B}}$$

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The peculiarities of vector equality only mean that this shows that the curves $\boldsymbol{\beta}$ and $F(\boldsymbol{\alpha})$ are parallel.

But by the remark above we do have $\boldsymbol{\beta} = F(\boldsymbol{\alpha})$.

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