1. Orientation of Frames (3d)

We've established in 3d that $\{\vec{T}, \vec{N}, \vec{B}\}$ form an orthonormal "frame." Any vector in 3d can be written as a linear combination of these three.

(1) Do $\{\vec{T}, \vec{N}, -\vec{B}\}$ also form such a frame?
(2) Do $\{\vec{T}, -\vec{N}, \vec{B}\}$?
(3) Do $\{-\vec{T}, -\vec{N}, \vec{B}\}$?
(4) Do $\{-\vec{T}, \vec{N}, -\vec{B}\}$?

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Since $\vec{B} \cdot \vec{B} =$ ____, we know that $\vec{B} \cdot \vec{T} \times \vec{N} = 1$.

(5) What about $(-\vec{B}) \cdot \vec{T} \times \vec{N}$ or $\vec{B} \cdot \vec{T} \times (-\vec{N})$ or $(-\vec{B}) \cdot (-\vec{T}) \times (-\vec{N})$? ____________?

(6) What about $(\vec{B}) \cdot (-\vec{T}) \times (-\vec{N})$ or $(-\vec{B}) \cdot (-\vec{T}) \times (\vec{N})$? ________

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Consider a frame $\{\vec{E}_1, \vec{E}_2, \vec{E}_3\}$.

It is called ______________ if

$$\vec{E}_1 \cdot \vec{E}_2 \times \vec{E}_3 = \_\_,$$

and ______________ if

$$\vec{E}_1 \cdot \vec{E}_2 \times \vec{E}_3 = \_\_\_.$$

(We are trying to distinguish between right- and left-"handedness.")

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2. Mappings and their Derivatives

We define a function $F : \mathbb{R}^n \to \mathbb{R}^m$ as a set of real-valued functions $\{f_1, f_2, \ldots, f_m\}$, such that

$$F(\vec{V}) = \Big(f_1(\vec{V}), f_2(\vec{V}), \ldots, f_m(\vec{V})\Big),$$

for any $\vec{V} \in \mathbb{R}^n$.

If $n = 1$, $F : \mathbb{R} \to \mathbb{R}^m$ defines a __________

If $m = 1$, F assigns a real number to each $\vec{V} \in \mathbb{R}^n$.

Example: ____

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We call F a ____________ if it is differentiable; i.e., if each of $f_1, f_2, \ldots, f_m$ are differentiable.

Now, we've often played fast and loose, whenever it was safe, with the notions of a *point* in $\mathbb{R}^n$, and a *vector* in $\mathbb{R}^n$.

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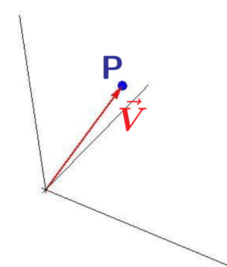
Below, the point **P** has the same *coordinates* as the vector $\vec{V}$ has *components*, say (a, b, c).

We've switched between viewing (a, b, c) as a point or a vector, based on convenience.



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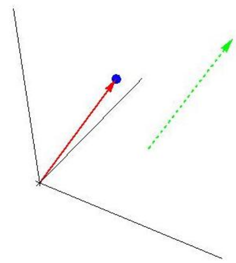
But, if we want a vector with the same components, but at a different location, we have to find a way of specifying this.

We'll use the notation $\vec{V}_{\mathbf{P}}$ to denote a vector located at a point **P**.



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We define the derivative mapping, F_*, of $F : \mathbb{R}^n \to \mathbb{R}^m$ as follows:

Let $\vec{V}_{\mathbf{P}}$ be a vector at $\mathbf{P} \in \mathbb{R}^n$. Consider the curve

$$\boldsymbol{\alpha}(t) = \mathbf{P} + t\vec{V}_{\mathbf{P}}$$

Let $$\boldsymbol{\beta}(t) = F(\boldsymbol{\alpha}(t))$$

be the image curve in $\mathbb{R}^m$ of $\boldsymbol{\alpha}$ in $\mathbb{R}^n$.

Define $$F_*(\vec{V}_{\mathbf{P}}) = \boldsymbol{\beta}'(0).$$

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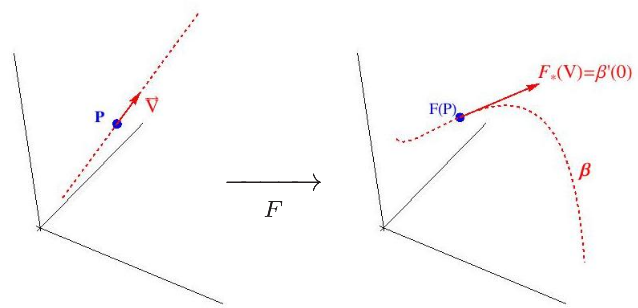
$\boldsymbol{\alpha}$ is the straight line in $\mathbb{R}^n$ through **P** in the direction of $\vec{V}_{\mathbf{P}}$; $\boldsymbol{\beta}$ is its image in $\mathbb{R}^m$ under F.



9 $\mathbb{R}^n$ $\mathbb{R}^m$

3. Isometries

Define the distance between two points

$P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$

in $\mathbb{R}^n$ by the usual Pythagorean formula:

If we view P, Q as vectors, $d(\vec{P}, \vec{Q}\,) =$

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We call $\mathbb{R}^n$ with this notion of distance ________ ____________

An isometry of $\mathbb{R}^n$ is a distance-preserving "mapping" (differentiable) ; i.e.,

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

has the property that

11 $$d(F(P), F(Q)) = d(P, Q) \quad \forall P,\, Q \in \mathbb{R}^n$$

Example 1: Translation

Think of the points of $\mathbb{R}^n$ as vectors, and let $\vec{A}$ be some fixed vector. We define a translation as the mapping given by

$$T(\vec{P}) = \vec{P} + \vec{A}\,, \quad \forall \vec{P} \in \mathbb{R}^n$$

(7) Is this is an isometry?

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Translations have additional properties: they are *injective* (distinct elements get mapped to distinct elements), and *surjective* (every element in $\mathbb{R}^n$ is mapped onto by some element).

As a result every translation, T, has an inverse translation, T^{-1}:

$$T(\vec{\boldsymbol{P}}) = \vec{\boldsymbol{P}} + \vec{\boldsymbol{A}} \qquad T^{-1}(\vec{\boldsymbol{P}}) =$$

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Example 2: Rotation (2d)

Here it's more useful to think of elements of $\mathbb{R}^2$ as ordered pairs of real numbers. A rotation, C, through an angle θ maps the point (p_1, p_2) to the point $(q_1, q_2) = C(p_1, p_2)$ given by

$$q_1 = p_1 \cos\theta - p_2 \sin\theta$$
$$q_2 = p_1 \sin\theta + p_2 \cos\theta$$

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(**8**) Is this is an isometry?

$q_1^2 + q_2^2 =$

Let's switch back to vectors.
Consider $\vec{\boldsymbol{V}} = (v_1, v_2)$ and $\vec{\boldsymbol{W}} = (w_1, w_2)$.

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It's easy to check that

- $C(\vec{\boldsymbol{V}} + \vec{\boldsymbol{W}}) =$ ____________________
- $C(\vec{\boldsymbol{x}}\, V) =$ ______________ for any scalar x.

This means that rotations are

____

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(**9**) Calculate $C(\vec{\boldsymbol{V}}) \cdot C(\vec{\boldsymbol{W}})$.

$C(\vec{\boldsymbol{V}}) =$

$C(\vec{\boldsymbol{W}}) =$

So $C(\vec{\boldsymbol{V}}) \cdot C(\vec{\boldsymbol{W}}) =$

$+$

$= v_1 w_1 + v_2 w_2 = \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{W}}$

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So, rotations preserve dot products.

A linear transformation that preserves dot products is called an ____________________

We show that every isometry can be decomposed into an orthogonal transformation followed by a translation.

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First: if F is an isometry such that $F(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$, then F is orthogonal.

1) F preserves norms.

$$\|F(\vec{\boldsymbol{V}})\| = d\big(F(\vec{\boldsymbol{V}}), \quad\big) = d\big(F(\vec{\boldsymbol{V}}), \qquad\big)$$
$$= \qquad =$$

2) F preserves dot products. We'll use

$d^2(\vec{\boldsymbol{V}}, \vec{\boldsymbol{W}}) = \|\vec{\boldsymbol{V}} - \vec{\boldsymbol{W}}\|^2 = (\vec{\boldsymbol{V}} - \vec{\boldsymbol{W}}) \cdot (\vec{\boldsymbol{V}} - \vec{\boldsymbol{W}})$.

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Since $d(F(\vec{\boldsymbol{V}}), F(\vec{\boldsymbol{W}})) = d(\vec{\boldsymbol{V}}, \vec{\boldsymbol{W}})$,

$$\big(F(\vec{\boldsymbol{V}}) - F(\vec{\boldsymbol{W}})\big) \cdot \big(F(\vec{\boldsymbol{V}}) - F(\vec{\boldsymbol{W}})\big) = (\vec{\boldsymbol{V}} - \vec{\boldsymbol{W}}) \cdot (\vec{\boldsymbol{V}} - \vec{\boldsymbol{W}})$$

or

$$\|F(\vec{\boldsymbol{V}})\|^2 - 2F(\vec{\boldsymbol{V}}) \cdot F(\vec{\boldsymbol{W}}) + \|F(\vec{\boldsymbol{W}})\|^2 = \|\vec{\boldsymbol{V}}\|^2 - 2\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{W}} + \|\vec{\boldsymbol{W}}\|^2$$

So, $F(\vec{\boldsymbol{V}}) \cdot F(\vec{\boldsymbol{W}}) = \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{W}}$.

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3) F is linear; i.e.,

$F(\vec{\boldsymbol{V}}+\vec{\boldsymbol{W}}) = F(\vec{\boldsymbol{V}})+F(\vec{\boldsymbol{W}})$ & $F(x\vec{\boldsymbol{V}}) = xF(\vec{\boldsymbol{V}})$.

(Following proof by Evangelos Taratoris, an MIT student.)

Let $\vec{\boldsymbol{U}} \equiv \vec{\boldsymbol{V}} + \vec{\boldsymbol{W}}$, and $\hat{\vec{\boldsymbol{V}}} = F(\vec{\boldsymbol{V}})$. Then

$$(\hat{\vec{\boldsymbol{U}}} - \hat{\vec{\boldsymbol{V}}} - \hat{\vec{\boldsymbol{W}}}) \cdot (\hat{\vec{\boldsymbol{U}}} - \hat{\vec{\boldsymbol{V}}} - \hat{\vec{\boldsymbol{W}}})$$
$$=$$

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Since F preserves dot products, this

$$=$$

$$=$$

$$= \quad .$$

Therefore $\hat{\vec{\boldsymbol{U}}} - \hat{\vec{\boldsymbol{V}}} - \hat{\vec{\boldsymbol{W}}} = \vec{\mathbf{0}}$. Or

$$F(\vec{\boldsymbol{V}} + \vec{\boldsymbol{W}}) = F(\vec{\boldsymbol{V}}) + F(\vec{\boldsymbol{W}})$$

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Exercise:

Show that $F(x\vec{\boldsymbol{V}}) = xF(\vec{\boldsymbol{V}}) \quad \forall x \in \mathbb{R}, \vec{\boldsymbol{V}} \in \mathbb{R}^n$.

Steps 1–3 show that if F is an isometry such that $F(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$, then it is orthogonal.

We use this to show that an arbitrary isometry F obeys $F = TC$, where juxtaposition is function composition, T is some translation and C some orthogonal transformation.

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Let T be translation by $F(\vec{\mathbf{0}})$: $T(\vec{\boldsymbol{V}}) = \vec{\boldsymbol{V}} + F(\vec{\mathbf{0}})$.

Then, T^{-1} is translation by ________.

We have

$$\big(T^{-1}F\big)(\vec{\mathbf{0}}) = T^{-1}\big(F(\vec{\mathbf{0}})\big) = F(\vec{\mathbf{0}}) - F(\vec{\mathbf{0}}) = \vec{\mathbf{0}}.$$

So $T^{-1}F$ is orthogonal, say C. In other words

$$F = TC.$$

It can be shown that this decomposition of F is unique. We'll call C, the orthogonal part of F.

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4. Congruence of Curves

We want to show that a curve in $\mathbb{R}^3$ is completely defined by its curvature, k, and torsion, τ.

In other words, if two curves have the same k and τ, they are "the same."

Clearly this needs some qualification.

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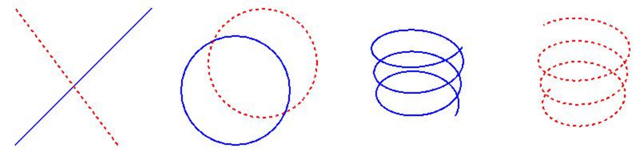
Do we regard the solid curves and the dashed curves of each shape below as being "the same" or different?



The curves in each pair are clearly different in location or orientation, but are otherwise "the same."

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We say that two curves,

$$\boldsymbol{\alpha},\ \boldsymbol{\beta} : I \to \mathbb{R}^3,$$

are ________________ if there exists an isometry F of $\mathbb{R}^3$ such that

We'll show that if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are curves with unit tangents, such that $k_\alpha = k_\beta$ and $\tau_\alpha = \pm\tau_\beta$, then they're congruent.

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