Arvind Borde / MTH 675, Unit 5: Curves and Curvature III

1. The Frenet-Serret Equations

Sticking with 3d, the three vectors, $\vec{T}$, $\vec{N}$ and $\vec{B}$, form an "orthonormal basis:" Any vector can be written as a linear combination of them.

(1) Why are $\vec{T}$ and $\vec{N}$ orthogonal?

____

____

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(2) Why is $\vec{B}$ orthogonal to $\vec{T}$ and $\vec{N}$?

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(3) Why are they unit?

$\vec{T}$: ________________________________

$\vec{N}$: ________________________

$\vec{B}$: ________________________

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From a direct calculation with components, one can show for any two vectors, $\vec{V}$ and $\vec{W}$ that

$$\|\vec{V}\times\vec{W}\|^2 = (\vec{V}\cdot\vec{V})(\vec{W}\cdot\vec{W}) - (\vec{V}\cdot\vec{W})^2$$

Applying this to $\vec{B} = \vec{T}\times\vec{N}$, we get directly that $\|\vec{B}\| = 1$.

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The Frenet-Serret formulas express the derivatives of $\vec{T}$, $\vec{N}$ and $\vec{B}$ in terms of these vectors themselves:

$$\vec{T}' =$$
$$\vec{N}' =$$
$$\vec{B}' =$$

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The first of these is ______________________

The last follows because __________________

____

The middle one needs showing. Let

$$\vec{N}' = x\vec{T} + y\vec{N} + z\vec{B}$$

5 Then $x = \vec{N}'\cdot\vec{T}$, $y = \vec{N}'\cdot\vec{N}$, and $z = \vec{N}'\cdot\vec{B}$.

(4) Differentiate $\vec{N}\cdot\vec{T} = 0$ to get x.

(5) y is zero. Why? ________________

(6) Differentiate $\vec{N}\cdot\vec{B} = 0$ to get z.

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ADDITIONAL NOTES

Expressing things in a frame defined by $\vec{T}$, $\vec{N}$ and $\vec{B}$, is often useful.

For example, consider a sphere of radius R.
(7) What would you guess is the least curved curve that you can draw on it? ________________

(8) What is its curvature? __________

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Let α be any curve on that sphere.
(9) What is $\alpha \cdot \alpha$? __________

(10) What does $(\alpha \cdot \alpha)'$ tell us?

(11) Differentiate *that* and use a S-F formula:

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Now, by the Schwarz inequality

$$|\alpha \cdot \vec{N}| \leqslant \|\alpha\| \|\vec{N}\| =$$

So

$$k = |k| = \frac{1}{|\alpha \cdot \vec{N}|} \geqslant$$

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2. An Example with a Moral

Das zweidimensionale Sinuskurve
We usually think of this geometrical "shape" (set of points in $\mathbb{R}^2$) as being defined by $y =$ ______

We can write this as a parametrized function:

$$\beta(t) = (\underset{x \searrow}{\quad}, \underset{\swarrow y}{\quad})$$

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(12) What's $\beta'(t)$? __________

(13) What's the arc length?

$$s(t) =$$

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This integral can't be done with commonly known functions (we need elliptic integrals).

So we need to work around this.

OK, you're questioning people.
(14) What question comes to mind?

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ADDITIONAL NOTES

Why?

To use arc length as the curve parameter, *so as to get the unit tangent to the curve*, and from there, the curvature, etc.

These are the steps we've trod:

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- Start with a curve $\boldsymbol{\beta}(t)$, with t an arbitrary parameter.
- Calculate $s(t)$ and invert it to find $t(s)$.
- Define $\boldsymbol{\alpha}(s) \equiv \boldsymbol{\beta}\big(t(s)\big)$.
- Find $\vec{\boldsymbol{T}}(s) = \dfrac{d\boldsymbol{\alpha}(s)}{ds}$.

It turns out, though, we can get $\vec{\boldsymbol{T}}$ without knowing $t(s)$ explicitly.

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(**15**) Find ds/dt in our example.

$$ds/dt =$$

(**16**) Use the chain rule and the inverse function theorem to get $\vec{\boldsymbol{T}}$.

$$\vec{\boldsymbol{T}} = \frac{d\boldsymbol{\alpha}}{ds} =$$

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(**17**) Remembering how we went from $y = sin\,x$ to the parametrized version $\boldsymbol{\beta}(t) = (t, sin\,t)$, does the $(1, cos\,t)$ part of $\vec{\boldsymbol{T}}$ remind you of anything familiar? ____________________

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(**18**) OK, enough fun. Is $\vec{\boldsymbol{T}}$ unit?

$$\|\vec{\boldsymbol{T}}\| = \sqrt{\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}}$$

$$=$$

$$=$$

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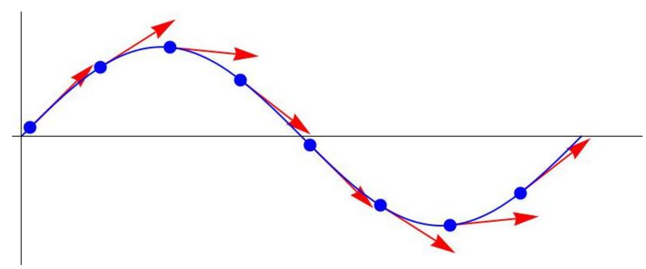
Graphically:



$\vec{\boldsymbol{T}}$ at various points

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ADDITIONAL NOTES

(**19**) With $\vec{\boldsymbol{T}} = \left(\frac{1}{\sqrt{1+cos^2(t)}}, \frac{cos\, t}{\sqrt{1+cos^2(t)}} \right)$, calculate

$$\frac{d\vec{\boldsymbol{T}}}{ds} =$$

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$$=$$

$$=$$

20

(**20**) Find $k(t)$.

$$k(t) = \|\vec{\boldsymbol{T}}'(t)\| =$$

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(**21**) What's

$k(0)$? ____

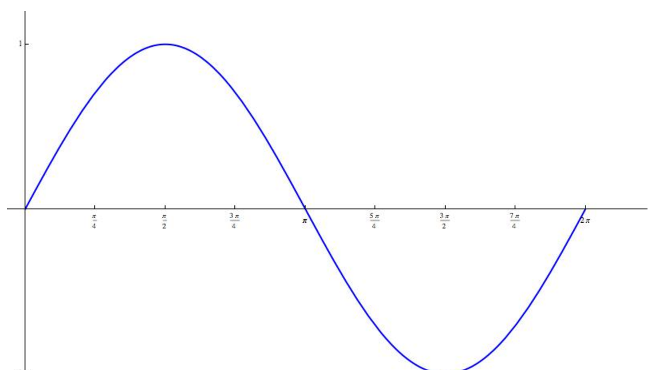
$k(\pi/4)$? ______________

$k(\pi/2)$? ____

$k(3\pi/4)$? ______________

$k(\pi)$? ____

22 Are these plausible? Let's look.



Where's the graph "straightest"?

23 Where's the graph "curviest"?

And, if you can believe Mathematica,

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{k(t)}$$

$$= \left(\left(1+cos^2 t\right) cot\, t \sqrt{\frac{sin^2 t}{(1+cos^2 t)^3}}, -\left(1+cos^2 t\right) csc\, t \sqrt{\frac{sin^2 t}{(1+cos^2 t)^3}} \right)$$

Coordinates of the center of the osculating circle:

$$\left(t + (1 + cos^2 t)\, cot\, t, -2\, cos\, t\, cot\, t\right)$$

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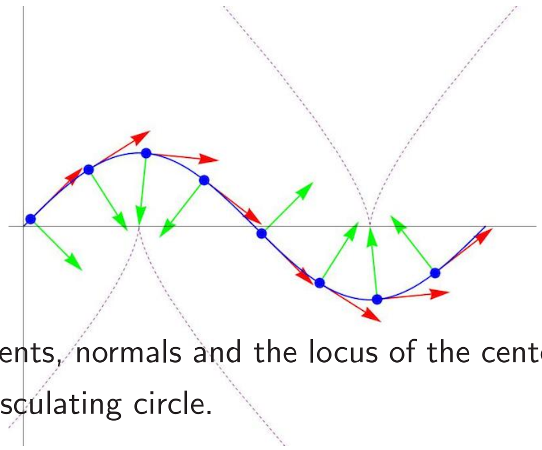
ADDITIONAL NOTES



Tangents, normals and the locus of the centers of the osculating circle.

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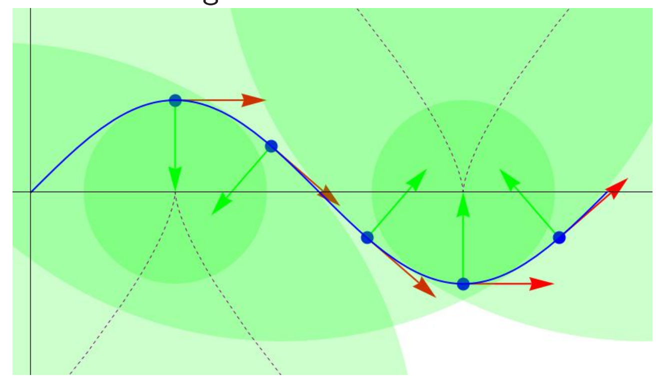
A few osculating circles:



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ADDITIONAL NOTES

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