Arvind Borde / MTH 675, Unit 4: Curves and Curvature II

1. Quick Review

Using arc length, s, along a curve $\boldsymbol{\alpha}(s)$ as the curve parameter leads to a derivative $\vec{\boldsymbol{T}}(s) \equiv \boldsymbol{\alpha}'(t)$ with unit norm:

$$\|\vec{\boldsymbol{T}}(s)\| \equiv \|\boldsymbol{\alpha}'(s)\| = 1$$

$\vec{\boldsymbol{T}}(s)$ is tangent to the curve.

We define the curvature as $k(s) = \|\vec{\boldsymbol{T}}'(s)\|$.

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For a circle of radius r in 2d, this yielded the believable result that

$$k(s) =$$

Consider another simple example: a straight line in 3d passing through the origin:

$$\boldsymbol{\beta}(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$$

Let's pick, for simplicity $t \in [0, 1]$.

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(1) What's an obvious way to parametrize it (i.e., write each $\beta_i(t)$ as some function of t)?

$\boldsymbol{\beta}(t) = (\underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}),$

(2) What's $\boldsymbol{\beta}'(t)$?

$$\boldsymbol{\beta}'(t) = (\underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}})$$

(3) What's $\|\boldsymbol{\beta}'(t)\|$?

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(4) What's the arc length?

$$s(t) =$$

(5) Reparametrize $\boldsymbol{\beta}(t)$ using arclength, s.

$$\boldsymbol{\alpha}(s) = (\underline{\hspace{2cm}}, \underline{\hspace{2cm}}, \underline{\hspace{2cm}})$$

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(6) What's $\vec{\boldsymbol{T}}(s) \equiv \boldsymbol{\alpha}'(s)$?

$$\vec{\boldsymbol{T}}(s) = (\underline{\hspace{2cm}}, \underline{\hspace{2cm}}, \underline{\hspace{2cm}})$$

(7) Is $\|\vec{\boldsymbol{T}}(s)\| = 1$? $\underline{\hspace{4cm}}$

(8) What's $\vec{\boldsymbol{T}}'(s)$? $\underline{\hspace{5cm}}$

Therefore, $k(s) \equiv \|\vec{\boldsymbol{T}}'(s)\| = \underline{\hspace{0.8cm}}$, as we'd expect.

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2. The Principal Normal Vector

(9) Using $\|\vec{\boldsymbol{T}}(s)\| = 1$, what's $\vec{\boldsymbol{T}}(s) \cdot \vec{\boldsymbol{T}}'(s)$?

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ADDITIONAL NOTES

If $\vec{T}'(s) \neq \vec{0}$, we define the ________________ ______________ as the unit vector in the direction of $\vec{T}'(s)$:

$$\vec{T}'(s) \neq \vec{0}$$

Or, $\vec{T}'(s) =$

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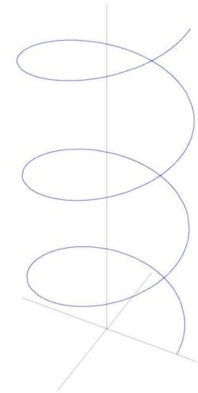
Example:
Consider a "circular helix" defined by
$\beta(t) = (a\,cos\,t, a\,sin\,t, bt)$;
a, b const.,
$t \in [0, B]$.



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(10) What's $\beta'(t)$?

$\beta'(t) =$

(11) What's $\|\beta'(t)\|$?

(12) What's the arc length?

$s(t) =$

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(13) Reparametrize the helix using the arc length, s, as the parameter:

$\alpha(s) =$

(14) Calculate $\vec{T}(s) = \alpha'(s)$ and $\vec{T}'(s)$.

$\vec{T}(s) =$

$\vec{T}'(s) =$

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(15) What's $\|\vec{T}(s)\|$? ___

(16) What's $k(s) = \|\vec{T}'(s)\|$? ______

So $\vec{T}'(s) = k(s)\vec{N}(s)$ where

$$\vec{N}(s) = (-\,cos(s/A), -\,sin(s/A), 0)$$

is clearly a unit vector.

Our helix, unsurprisingly, is a curve of constant curvature.

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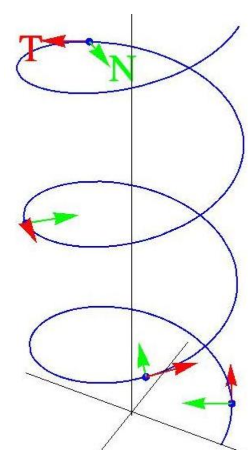
Here are unit tangents, $\vec{T}(s)$, and principal normals, $\vec{N}(s)$, drawn at several points on a helix.



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ADDITIONAL NOTES

3. Osculation (kissing)

Geometrically, the term refers to the contact of two curves at a point where they have a common tangent.

The ____________________ of a curve, $\boldsymbol{\alpha}(s)$, at a given s is the point $c(s)$ with the *coordinates of*

$$\boldsymbol{\alpha}(s) + \frac{\vec{\boldsymbol{N}}(s)}{k(s)}$$

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(**17**) Find $c(s)$, for any $s \in [a, b]$ for our helix, using $A^2 = a^2 + b^2$ to simplify the answer:

$$\boldsymbol{\alpha}(s) = (a\,cos(s/A), a\,sin(s/A), bs/A)$$

$$\frac{\vec{\boldsymbol{N}}(s)}{k(s)} =$$

$$c(s) =$$

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To get concrete,

(**18**) What are

$\boldsymbol{\alpha}(A\pi/2)$ ____________________

and

$c(A\pi/2)$? ________________________

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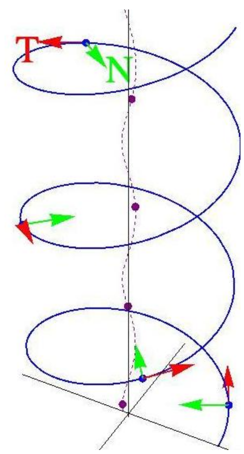
The dashed helix is the locus of the centers of curvatures of points on the original helix. $\vec{\boldsymbol{N}}(s)$ points to the "matching" $c(s)$.



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We call the plane that's defined by $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{N}}$ the

____

Where does the kissing come in?

The circle that lies in the osculating plane with center $c(s)$ and radius $1/k(s)$ will touch the curve at the point with coordinates $\boldsymbol{\alpha}(s)$, and will have the same tangent as the curve does at that point.

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This circle is called the ________________________

At any point, given by a parameter value s, the osculating circle has been engineered to have the same curvature $k(s)$ as the curve $\boldsymbol{\alpha}(s)$.

For it to touch the curve at the point in question, the length of the vector from that particular $\boldsymbol{\alpha}(s)$ to the point $c(s)$ has to be $1/k(s)$.

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ADDITIONAL NOTES

Playing fast and loose (but, with luck, not "lose") with coordinates and vectors, from the definition of $c(s)$, we have

$$\|c(s) - \boldsymbol{\alpha}(s)\| = \left\| \frac{\vec{\boldsymbol{N}}(s)}{k(s)} \right\| =$$

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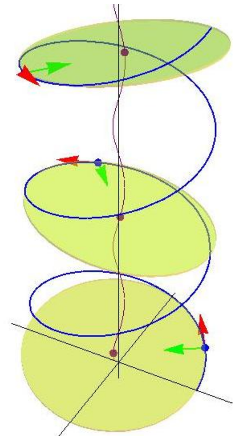
The osculating circle (filled in as a disk here) drawn at a few points on a helix, along with $\vec{\boldsymbol{T}}(s)$ (tangential) and $\vec{\boldsymbol{N}}(s)$ (radial).



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4. Binormal Vectors and Torsion

In general, an osculating plane tilts as it moves along a curve. Changes in a vector orthogonal to that plane, the ________, measures this:

$$\vec{\boldsymbol{B}}(s) = \vec{\boldsymbol{T}}(s) \qquad \vec{\boldsymbol{N}}(s)$$

We measure the gyrations of the osculating plane by computing ________.

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(19) With all the power of cross products and their derivatives at your command, calculate $\vec{\boldsymbol{B}}\,'$ (we take "(s)" for granted).

$$\vec{\boldsymbol{B}}\,' =$$

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So $\vec{\boldsymbol{B}}\,'$ is ________ to both $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{N}}\,'$.

So is $\vec{\boldsymbol{N}}$:

It's orthogonal to $\vec{\boldsymbol{N}}\,'$ because it's ________

It's orthogonal to $\vec{\boldsymbol{T}}$ because it's $\propto \vec{\boldsymbol{T}}\,'$ and $\vec{\boldsymbol{T}}$ is unit. In 3d, we must have

$$\vec{\boldsymbol{B}}\,' \propto \vec{\boldsymbol{N}} \quad \text{or} \quad \vec{\boldsymbol{B}}\,'(s) = -\tau(s)\vec{\boldsymbol{N}}(s).$$

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We call $\tau(s)$ the ________ of the curve. It's a measure of how much the osculating plane "tilts."

How does this work for our helix?

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ADDITIONAL NOTES

(20) Compute $\vec{B} = \vec{T} \times \vec{N}$ where (reminder)

$\vec{T}(s) = \frac{1}{A}(-a\,sin(s/A), a\,cos(a/A), b)$

$\vec{N}(s) = (-\,cos(s/A), -\,sin(s/A), 0)$

$\vec{B} = \vec{T} \times \vec{N}$

$= (T_2N_3 - T_3N_2, T_3N_1 - T_1N3, T_1N_2 - T_2N_1)$

$=$

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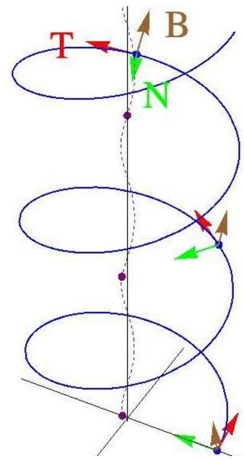
$\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ drawn at a few points on a helix, along with the matching centers of curvature.



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(21) Compute $\vec{B}'$.

$\vec{B}' =$

(22) Comparing with $\vec{N}$, what's $\tau(s)$ here?

$\tau(s) =$

(23) Does it seem plausible that the osculating plane "tilts" at a fixed rate for our helix? ____

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5. Review

Let $\alpha(s)$ be a curve parametrized by arc length.

- $\vec{T}(s) =$ ____
  G: ____
- $k(s) =$ ____
  G: ____
- $\vec{N}(s) =$ ____
  G: ____

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- $c(s) =$ ____
  G: ____
- $\vec{B}(s) =$ ____
  G: ____
- $\tau(s)$ ____
  G: ____

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ADDITIONAL NOTES

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