Arvind Borde / MTH 675, Unit 3: Curves and Curvature

1. Curves

A ____________ is a vector-valued function of a real variable, t, defined on an interval $[a, b]$:

We call t the ________________ of the curve.

We assume our curves are smooth – components are differentiable functions of the parameter.

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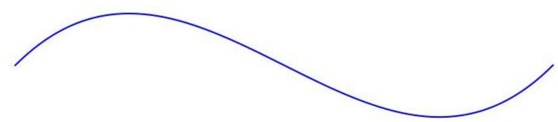
This definition says that a "curve" is a *function*. What we normally think of as a curve is the ____________ of the function. It's the set of points $\{(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)) \mid t \in [a, b]\}$.



Still, we'll sometimes loosely speak of a curve, such as the one shown, simply as $\boldsymbol{\alpha}$.

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2. Derivatives and Tangents

The _____________________ of a curve $\boldsymbol{\alpha}(t)$ is

where a prime is a derivative with respect to t.

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Let's get right to an example (2d):

$$\boldsymbol{\alpha}(t) = (r\,cos\,t, r\,sin\,t) \qquad t \in [0, 2\pi]$$

(1) What's $\boldsymbol{\alpha}'(t)$?

(2) What's $\boldsymbol{\alpha}'(t) \cdot \boldsymbol{\alpha}(t)$? __

(3) What does this say about $\boldsymbol{\alpha}'(t)$ and $\boldsymbol{\alpha}(t)$?

4 ________________________

Let's pick $r = 1$.

(4) What's $\boldsymbol{\alpha}(\pi/4)$?

(5) What's $\boldsymbol{\alpha}'(\pi/4)$?

(6) Are they orthogonal? ___________

5

What does this represent in the "vectors are arrows" picture?

Let's stick with $r = 1$.

(7) Plot $\boldsymbol{\alpha}(t) = (1\,cos\,t, 1\,sin\,t),\ t \in [0, 2\pi]$.
(Use $t = 0, \pi/4, \pi/2, \ldots, 2\pi$)

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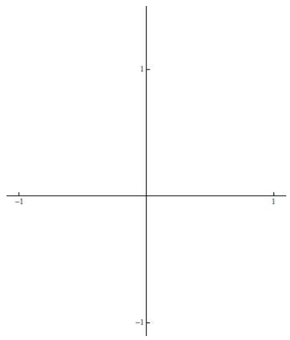
ADDITIONAL NOTES



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(8) Do these look orthogonal? ____________

(9) What's the geometrical relationship that $\boldsymbol{\alpha}'(t)$ appears to have to the image of $\boldsymbol{\alpha}(t)$ (the curve)?

____

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The geometrical interpretation of the derivative vector, is exactly what it is for images (graphs) of a real-valued function, $f(x)$.

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What is *that*? Analyze the expression

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The numerator here, $f(x+\Delta x)-f(x)$, is simply the _____________________, and may be thought of as ______________.

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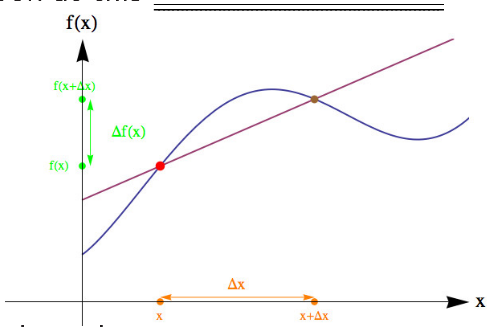
Now, look at this __________________



It goes through

11 ________ and ___________________.

(10) What is the slope of this secant?

(11) What aspect of straight line behavior does the "slope" capture? ______________________

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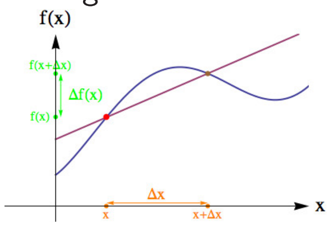
ADDITIONAL NOTES

So $\frac{\Delta f(x)}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$ captures the idea of the rate of change of the secant.



What about the rate of change of $f(x)$?

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The slope of the secant gives us a measure of the ______________________________ of $f(x)$ between x and $x + \Delta x$.

(12) If we want the rate of change *at* x, how might we do better?

____

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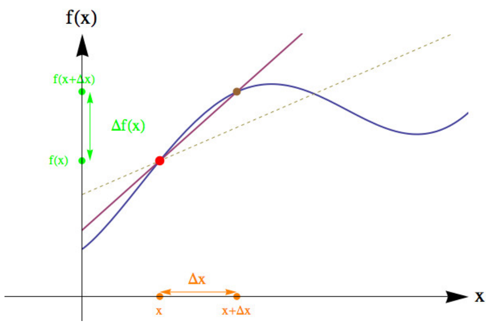
Smaller Δx...



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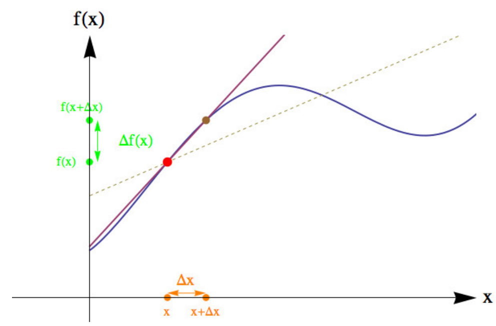
... and smaller ...



The slope of these secants gives a better and better measure of the rate of change of $f(x)$ at x.

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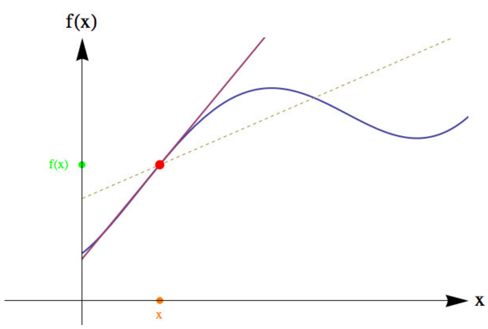
In the limit $\Delta x \to 0$, the derivative gives the ____



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For vector valued functions, too, the derivative of a curve at a point is the tangent vector to the image of the curve at that point.

For a slightly more complex example (3d), look at

$$\boldsymbol{\alpha}(t) = (t^{1/2} \cos t \sin t, \cos t, t/10)$$

(13) Draw the image and the derivative at several points.

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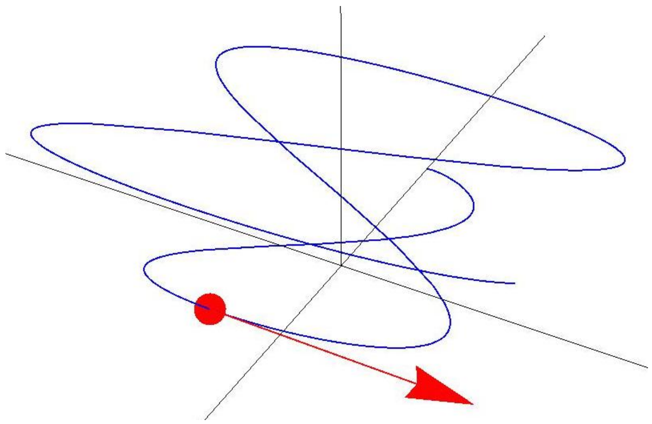
ADDITIONAL NOTES



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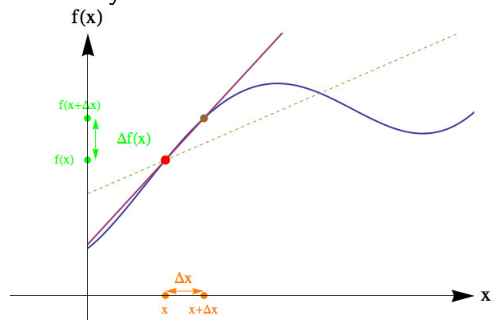
3. Arc (Curve) Length

We'll develop a formula for the length of a curve, s, motivated by



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Clearly,

$$\|\Delta\boldsymbol{\alpha}(t)\| \approx \Delta s$$

So

$$\frac{\|\Delta\boldsymbol{\alpha}(t)\|}{\Delta t} \approx \frac{\Delta s}{\Delta t}$$

or

$$\|\boldsymbol{\alpha}(t)'\| \approx \frac{\Delta s}{\Delta t}$$

or

$$\Delta s = \|\boldsymbol{\alpha}(t)'\|\Delta t$$

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All of which motivates the definition of arc length, s, of a curve, $\boldsymbol{\alpha}(t)$, defined on $[a, b]$, from a to t:

(14) What's the length of $\boldsymbol{\alpha}(t) = (r\, cos\, t, r\, sin\, t)$ from 0 to π?

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$$\boldsymbol{\alpha}'(t) =$$

So,

$$s =$$

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4. Arc Length As a Parameter

We've discussed

$$\boldsymbol{\alpha}(t) = (r\, cos\, t, r\, sin\, t) \qquad t \in [0, 2\pi]$$

(15) Would there be a difference between the images of that function that and this

$$\boldsymbol{\beta}(t) = (r\, cos\, 2t, r\, sin\, 2t) \qquad t \in [0, \pi]$$

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ADDITIONAL NOTES

It's useful to choose arc length, s, as a standard parameter.

This means that $\|\boldsymbol{\alpha}'(s)\| = 1$.

For a circle of radius r, this parametrization means

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5. Curvature

If $\vec{\boldsymbol{T}}(s) = \boldsymbol{\alpha}'(s)$, we define the curvature, $k(s)$, of a curve, $\boldsymbol{\alpha}(s)$, as

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(**16**) What's $\boldsymbol{\alpha}'(s)$ for a circle?

(**17**) What's $\boldsymbol{\alpha}''(s)$ for a circle?

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(**18**) What's $k(s) = \|\boldsymbol{\alpha}''(s)\|$ here?

(**19**) Does this seem reasonable? ____________

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ADDITIONAL NOTES

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