Arvind Borde / MTH 675, Unit 21: Finding Spacetime Metrics

-3. The Expansion of the Universe

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\,\theta d\phi^2)\right]$$

We've said that $\dot{a}(t) > 0$ is equivalent to saying that the Universe is expanding, but how does this connect to what we measure?

Note:

We call $H_0 \equiv \dot{a}(t)/a(t)$ Hubble's "constant."

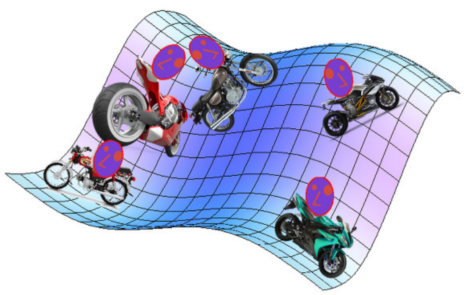
1

Consider observers separated by fixed coordinate distances r. The physical distance between them is $R(t) = a(t)r$.

Even if they have fixed coordinates, the distance between them changes because $a(t)$ does.

2

They're not moving away from everything else by actively running away:



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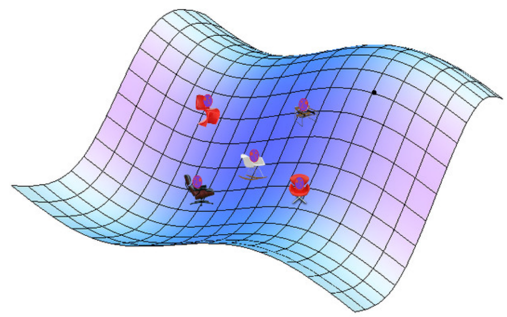
Instead, they're sitting in place



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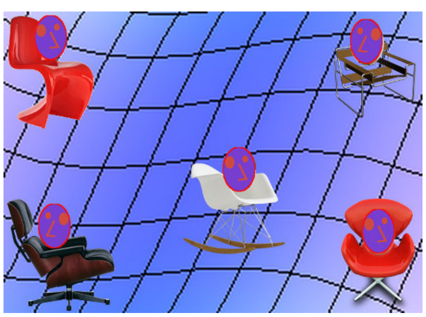
and enjoying at no charge the benefits of the expansion of the Universe:



5

The "hot big bang theory" makes testable predictions.

One is that there's remnant radiation running around the Universe at a temperature of $\sim 3°$K at microwave wavelengths: the CMBR.

Hunted by Dicke and Peebles at Princeton in the 1960s, but accidentally discovered as noise at Bell Labs by Penzias and Wilson.

6

ADDITIONAL NOTES

-2. The Critical Density

Remember: $$\frac{3\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{c^4}\rho - \frac{3k}{a^2(t)} \qquad (\diamond)$$

(1) Solve for ρ (and use $H_0 = \dot{a}(t)/a(t)$).

$\rho =$

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(2) Find a relation (equation or inequality) between ρ and $\frac{c^4}{8\pi G}(3H_0^2)$ when $k = -1,\ 0,\ 1$.

$k = -1$:

$k = 0$:

$k = 1$:

8

We call $\rho = \dfrac{c^4}{8\pi G}(3H_0^2)$ the critical density of the Universe, ρ_c. It's the dividing density between an open and a closed Universe, and between continual expansion and recollapse.

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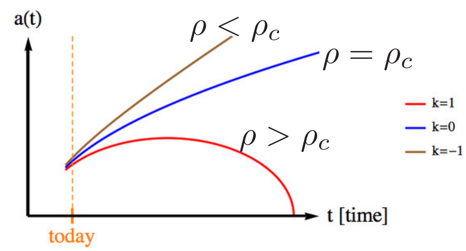

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That's what you'd expect on physical grounds. It takes matter to curve spacetime. If there's a low level of matter, you don't expect enough curvature to "close the Universe" or cause recollapse:

"Matter tells geometry how to curve."

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-1. Big Bang Problems

By the late 1970s, the big bang theory was considered on solid footing.

The origin of the uniformity of the Universe was unknown, but indisputable: the uniformity of CMBR was solid proof.

The density of the Universe was unknown, but it appeared to be close to ρ_c.

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ADDITIONAL NOTES

Then, at a conference in 1979 to celebrate the 100th anniversary of the birth of Albert Einstein, Dicke and Peebles (Drs. Microwave Background), discussed puzzling aspects of the big bang theory:

9. The big bang cosmology – enigmas and nostrums†

R. H. DICKE AND P. J. E. PEEBLES

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They said

The big bang cosmology that developed out of Einstein's ideas and Hubble's observations has stood the test of time and observation, but even the staunchest advocate would admit that it is at best only a reasonable first approximation that certainly does not tell the whole story. There are in particular some curious and enigmatic features of this cosmology that lead us to think that an important piece of the picture may be missing. It is useful to review and reconsider these curiosities from time to time because they certainly have something to teach us. But what is it?

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then went on with a quick account of what was understood about the Universe in 1917:

It is not clear how familiar Einstein was with the observational situation in astronomy, or how much attention he paid to it. There was at the time speculation that the spiral nebulae are island universes like the Milky Way galaxy, but there were also some good arguments that these objects must be only minor satellites. It was considered well established, from star counts, that the Milky Way star system is finite and bounded, shaped roughly like a flattened spheroid. The spiral nebulae seemed to be concentrated at the poles of this star system, which would suggest they are related to it. Also, by 1916 van Maanen had found the first tentative evidence of proper motions in some of the larger spirals (van Maanen, 1916). If valid, and if the internal velocities in these systems are less than the velocity of light, it would make them quite close and much smaller than the Milky Way.

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Nebulae were hazy patches that had been catalogued because they got in the way of finding the really interesting objects: comets.

The nature of these nebulae was debated. Some astronomers said they were in the Milky Way, others argued that they were outside it.

One argument that they were within the MW was the observations of novae in some of them.

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If a single star outside the MW could be observed brightening, the event would have to be "on a scale of magnitude such as the imagination recoils from contemplating."

"A Popular History of Astronomy during the Nineteenth Century," Agnes M. Clerke, page 438.

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The question relates to the size of the Universe: Is the MW the entire Universe, or does the Universe extend outside the MW?

The Milky Was was already known to be vast.

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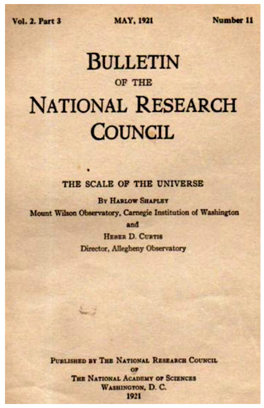
ADDITIONAL NOTES

The question was debated on 26 April 1920 by two leading astronomers.

The event became known as the Shapley-Curtis "Great Debate."



Vol. 2. Part 3 MAY, 1921 Number 11

BULLETIN OF THE NATIONAL RESEARCH COUNCIL

THE SCALE OF THE UNIVERSE

By Harlow Shapley
Mount Wilson Observatory, Carnegie Institution of Washington
and
Heber D. Curtis
Director, Allegheny Observatory

Published by The National Research Council of The National Academy of Sciences
Washington, D. C.
1921

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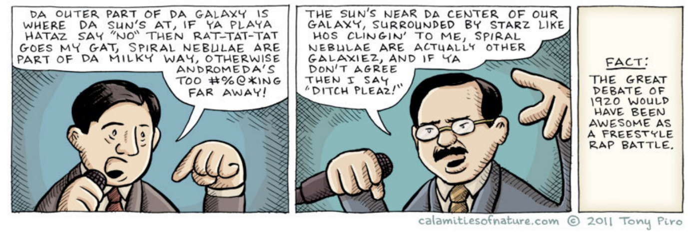
Here's the debate in your language:



The "within MW" view was thought to have won.

20

Although Einstein had been trying to build a larger cosmology, the indications were that the Milky Way was the entire Universe. But:

> Over the next two decades it became clear that these indications are misleading, the former because interstellar dust in the plane of the Milky Way obscures the galaxies and the latter because of observational problems. Einstein's vision was remarkably good. In 1924 Hubble showed, by the identification of variable stars of known intrinsic luminosity, that the spiral nebulae are well outside the Milky Way and at least comparable to it in size (Hubble, 1924). Hubble's surveys of the galaxy distribution, begun in 1926 (Hubble, 1926) and continuing through the 1930s, gave the first direct evidence of large-scale homogeneity and isotropy. This has been confirmed by recent observations of the precise isotropy of the radiation background – X-ray, microwave,

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The observations that the Universe was much bigger than we'd originally thought had also provided evidence of large scale homogeneity and isotropy. This is remarkable:

> The concept of large-scale homogeneity has been with us so long that cosmologists tend to take it as a commonplace, but it is remarkable simply because it stands in such contrast to our experience that things have structure – from the properties of subatomic particles on up to the organization of galaxies in great clouds. Milne (1935) was the first to

22

It's not only remarkable, large scale homogeneity is a problem:

> The distant galaxies observed in well-separated parts of the sky are so far apart from each other that there is not time enough since the big bang for a signal to have traveled from one to the other. Observers on Earth can see and compare them, being about half-way in between, and in line with homogeneity it is found that the galaxies are quite similar. By comparing radiation background intensities across the sky it is also found that the temperature and expansion rate are precisely synchronized across the visible universe. Even though the separate parts of the visible universe are not visible to each other they are evolving in very precise unison.

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We believe the Universe is $\sim$ 14 billion years old.

A patch of the Universe 12 billion light years from us in one direction cannot have communicated with another patch 12 billion light years from us in the opposite direction.

Yet they look virtually identical. What made them that way, if there was no communication between them?

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ADDITIONAL NOTES

Is this really a problem?

> Are structural relations between widely separated parts of the universe a problem? In the past these parts were much closer together. But close proximity in earlier times does not eliminate the problem. Assum-

This problem, the problem of inexplicable uniformity, is known as the

____

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Whether you think it a problem depends on how content you are with coincidence.

OK, we have a problem. Actually, we have two.

Dicke and Peebles, go right on to say:

> The relationships of widely separated parts of the universe are not the only problem. There is a remarkable balance of mass density and expansion rate. In general relativity theory with $\Lambda = 0$ the two are

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> related by the equation
>
> $$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \tfrac{8}{3}\pi G\rho(t) - \frac{c^2}{R^2a^2}, \tag{9.1}$$
>
> where $a(t)$ is the expansion parameter, R is a constant, and $|R|a(t)$ is the magnitude of the space curvature (measured in a hypersurface of roughly constant galaxy proper number density, at fixed cosmic time t). The present relative value of the two terms on the right side of this equation is poorly known, because the mean mass density, ρ, is so uncertain, but it is unlikely that the first term is less than 3 per cent of the magnitude of the second. Since ρ varies as a^{-3} (or more rapidly if pressure is important) the mass term on the right-hand side dominates the curvature term when a is less than about 3 per cent of its present value. Tracing the expansion back in time, one finds that at $t \sim 1$ s, when much of the helium is thought to have been produced, the mass term is some 14 orders of magnitude larger than the curvature term. This means the expansion rate has been tuned to agree with the mass density to an accuracy better than 1 part in 10^{14}. In the limit $t \to 0$ this balance

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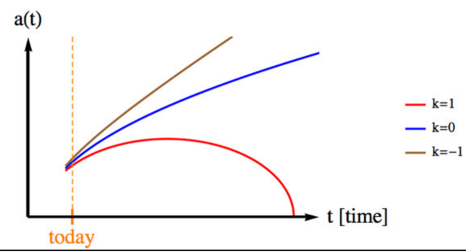
Remember, $k = 0$ gives the critical density, the dividing density between

a closed Universe ($k = 1, \rho > \rho_c$) and

an open Universe ($k = -1, \rho < \rho_c$).



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In Slide 7, from ($\diamondsuit$) we'd obtained:

$$\rho = \frac{c^4}{8\pi G}\left(3H_0^2 + \frac{3k}{a^2(t)}\right) = \frac{c^4}{8\pi G}(3H_0^2) + \frac{c^4}{8\pi G}\frac{3k}{a^2(t)}.$$

Writing the right-hand side in terms of ρ_c:

$$\rho = \rho_c + \frac{3kc^4}{8\pi G}\frac{1}{a^2(t)}$$

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Solving for ρ_c:

$$\rho_c = \rho - \frac{3kc^4}{8\pi G}\frac{1}{a^2(t)}$$

(3) What is $\rho_c/\rho - 1$?

$$\frac{\rho_c}{\rho} - 1 = -\frac{3kc^4}{8\pi G}\frac{1}{\rho a^2(t)}$$

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ADDITIONAL NOTES

The quantity ρ/ρ_c is called the density parameter and denoted by Ω . So we have

$$(\Omega^{-1} - 1)\rho a^2(t) = -\frac{3kc^4}{8\pi G} = \text{ constant.}$$

Writing $\rho a^2(t)$ as $\rho a^3(t)/a(t)$, and arguing that $\rho a^3(t)$ is roughly fixed, $\rho a^2(t)$ goes down (rapidly) as the Universe expands.

Therefore, $(\Omega^{-1} - 1)$ must go up rapidly.

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Why is that a problem?

The observed density of the Universe is estimated to be between a tenth and twice the critical density of about 10^{-30} g/cc: i.e., $0.1 < \Omega < 2$.

(4) What are the bounds on $(\Omega^{-1} - 1)$?

32

For a rapidly growing quantity to be this small today (after 14 billion years of growth), the density parameter $\Omega = \rho/\rho_c$ would have had to have been between 0.99999999999999 and 1.00000000000001 1 sec after the big bang.

> the expansion rate has been tuned to agree with the mass density to an accuracy better than 1 part in 10^{14}. In the limit $t \to 0$ this balance

33

Even small deviations from this cause the Universe to behave very differently from what we see.

A slightly greater value for ρ would make the Universe to be closed, and recollapse rapidly. A slightly smaller value would give too rapid an expansion.

Why the Universe should be so precisely tuned to the critical (i.e, flat) value of the density is called the______________.

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Dicke and Peebles presented the horizon and flatness problems at the Einstein centenary conference in 1979, but had already been giving lectures about them at other venues.

One lecture was by Dicke at Cornell University on November 13, 1978, on the flatness problem.

It was attended by a young particle physicist, Alan Guth.

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Guth got his Ph.D. from MIT in 1972 and had hopped around in temporary positions since:

- Princeton, 1971–74
- Columbia, 1974–76
- Cornell, 1976–79
- SLAC, 1979–1980

In 1978 Guth felt cosmology wasn't interesting. He went to Dicke's lecture because Penzias & Wilson had just received a Nobel Prize.

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ADDITIONAL NOTES

Guth was working on "Grand Unified Theory" at the time – the theory that unifies the strong nuclear, weak nuclear and electromagnetic forces at high energies (temperatures).

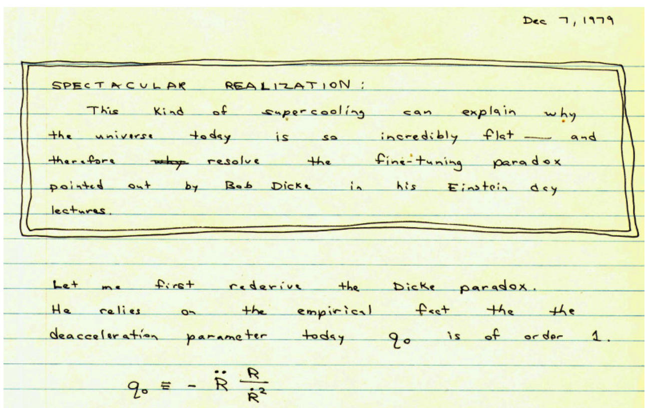
In 1970, we'd had a good understanding of just electromagnetism (and gravity).

By 1976, we had a unified theory of three of the
37 four fundamental forces.

Guth was investigating the existence of magnetic monopoles in GUT: Could they have existed in the early Universe and, if they did, why might we not see them today?

On the night of December 6, 1979, Guth felt he had found the reason why we see no monopoles today, through a mechanism called supercooling.

38 Then he remembered Dicke's talk (a year ago)...



Dec 7, 1979

SPECTACULAR REALIZATION:
This kind of supercooling can explain why the universe today is so incredibly flat — and therefore ~~why~~ resolve the fine-tuning paradox pointed out by Bob Dicke in his Einstein day lectures.

Let me first rederive the Dicke paradox.
He relies on the empirical fact the the deacceleration parameter today q_0 is of order 1.

$$q_0 \equiv - \ddot{R} \frac{R}{\dot{R}^2}$$

39

Later that same month, Guth was told over lunch about the horizon problem of cosmology (he didn't know of it at the time). He went home and figured out, that afternoon itself, that his "supercooling" mechanism would solve this problem too.

It was, as he said later, as if he'd found the master key to the Universe: door after door opened with
40 that single key.

Guth's career after December 1979:

- January 23, 1980: Announces his ideas in a seminar at SLAC
- January 24, 1980, before lunch: gets invitations to present his ideas from three different universities.
- January 24, 1980, after lunch: Is invited to spend three further years at SLAC; hears that U. Penn, and UC Davis are considering offering permanent professorships.
- January 28, 1980: U. Penn offers the job.
- February-March, 1980: Lectures at ten universities, including Harvard, Princeton, Columbia and Cornell.

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- On his return has offers of professorships from Minnesota, Rutgers, Harvard, Princeton, Maryland, UC Davis and UC Santa Barbara.
- But he wants MIT – and shortly after, he gets it. He's been there ever since.

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ADDITIONAL NOTES

What was this big idea of Alan Guth's?

Inflationary universe: A possible solution to the horizon and flatness problems

Alan H. Guth*

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(Received 11 August 1980)

The standard model of hot big-bang cosmology requires initial conditions which are problematic in two ways: (1) The early universe is assumed to be highly homogeneous, in spite of the fact that separated regions were causally disconnected (horizon problem); and (2) the initial value of the Hubble constant must be fine tuned to extraordinary accuracy to produce a universe as flat (i.e., near critical mass density) as the one we see today (flatness problem). These problems would disappear if, in its early history, the universe supercooled to temperatures 28 or more orders of magnitude below the critical temperature for some phase transition. A huge expansion factor would then result from a period of exponential growth, and the entropy of the universe would be multiplied by a huge factor when the latent heat is released. Such a scenario is completely natural in the context of grand unified models of elementary-particle interactions. In such models, the supercooling is also relevant to the problem of monopole suppression. Unfortunately, the scenario seems to lead to some unacceptable consequences, so modifications must be sought.

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0. Milne's Universe

Assume that $P = 0$ in the FLRW equations

$$\frac{3\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{c^4}\rho - \frac{3k}{a^2(t)}$$

$$\frac{3\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{c^4}(\rho + 3P)$$

44

What are the possibilities as $\rho \to 0$?

Setting $P = 0$:

$$\frac{3\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{c^4}\rho - \frac{3k}{a^2(t)}$$

$$\frac{3\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{c^4}\rho$$

What happens in these equations as $\rho \to 0$?

45

$$\dot{a}^2(t) \approx -k \quad \text{and} \quad \ddot{a}(t) \approx 0$$

with $k = 0$ or -1.

The second equation simply confirms the first, and has no further content. So we're working with

as our two cases.

46

Case 1: $k = 0$, $\therefore\ \dot{a}(t) \approx 0$.

If we scale the FLRW r coordinate to $\hat{r} = ar$, we get $d\hat{r} = adr$ and the metric becomes

47

Case 2: $k = -1$, $\therefore\ \dot{a}(t) \approx \pm 1$.

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ADDITIONAL NOTES

We've rediscovered "Milne's Universe."

(Wikipedia sagely observes "the assumption of zero energy content limits its use as a realistic description of the universe.")

But, it's thrilling.

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1 Spacelikeness and Timelikeness

For a vector $\vec{\boldsymbol{V}}$, we've defined

$$\vec{\boldsymbol{V}} \text{ is } \begin{cases} \text{timelike} & \text{if } g_{ab}V^aV^b > 0, \\ \text{null} & \text{if } g_{ab}V^aV^b = 0, \\ \text{spacelike} & \text{if } g_{ab}V^aV^b < 0, \end{cases}$$

where $g_{ab}V^aV^b$ is $g(\vec{\boldsymbol{V}}, \vec{\boldsymbol{V}})$ if you're working abstractly, or $g_{ij}V^iV^j$ if thinking concretely in components. In the same spirit, we'll use V^a for $\vec{\boldsymbol{V}}$.

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A (smooth) curve represents the path of:

a) a material test particle (such as a neutron), if its tangent is everywhere timelike.

b) a massless test particle (such as a photon), if its tangent is everywhere null.

If no (other) forces are at play, these paths are geodesic.

51

Let $f : \mathcal{M} \to \mathbb{R}$ be a scalar function on an nd spacetime, $\mathcal{M}$. Define a (local) hypersurface, $\mathcal{H}$, as the $(n-1)$d submanifold given by $f =$ constant.

The normal to $\mathcal{H}$ is obtained from the gradient of f, calculated in coordinates $\{x^i\}$ as $N_i = \{\partial_i f\}$, where $\partial_i f \equiv \partial f / \partial x^i$.

We define $N^i \equiv g^{ij} N_j$.

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$$\mathcal{H} \text{ is called } \begin{cases} \text{spacelike} & \text{if } N^i \text{ is timelike,} \\ \text{null} & \text{if } N^i \text{ is null,} \\ \text{timelike} & \text{if } N^i \text{ is spacelike} \end{cases}$$

For example, in 4d Minkowski spacetime with coordinates (t, x, y, z), t is a scalar function whose level hypersurfaces are spacelike ($N^i = (1, 0, 0, 0)$).

A spacelike hypersurface is "an instant of time," or, as Kurt Gödel put it, "a layer of 'now'."

53

Gödel came up with his characterization in an attempt to answer whether "reality" can always be split into layers of 'now.'

The answer he came up with was, in general, no.

In cosmology we start with the simplifying assumption that it can.

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ADDITIONAL NOTES

2. Symmetries

Consider a one-to-one mapping of a manifold onto itself: $\phi : \mathcal{M} \to \mathcal{M}$.

The mapping is called a homeomorphism, as we've seen, if it and ϕ^{-1} are continuous.

It is called a diffeomorphism if it and ϕ^{-1} are smooth.

55

Q What's a function, $f(x) : \mathbb{R} \to \mathbb{R}$, that's smooth, but whose inverse is not? ________________

56

What does $\phi : \mathcal{M} \to \mathcal{M}$ do to other entities?

Scalar function f: $\phi(f(P)) = f(\phi(P))$.

Vectors: Along the lines of a "derivative mapping" (U6), the mapping ϕ defines a mapping, ϕ^*, that carries vectors at a point $P \in \mathcal{M}$ to vectors at the point $\phi(P) \in \mathcal{M}$.

Metric: ϕ carries the metric at P to one at $\phi(P)$.

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If the mapped metric is the same as the original at $\phi(P)$, we call the mapping an isometry.

This "metric preserving" mapping, generalizes the idea of an isometry as a "distance preserving mapping" (U6).

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3. The Form of the FLRW Metric

We look at a 4d spacetime and assume there's "time coordinate" t and a set of "spatial coordinates," such that

a) $t =$ constant hypersurfaces, $\mathcal{H}$, do not intersect each other and are (going to be) spacelike;

b) fixed values of the spatial coordinates define timeline lines in the "t direction."

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We express the idea of spatial homogeneity by saying that for any $p, q \in \mathcal{H}$, there's an isometry that maps p to q.

Expressing isotropy mathematically is slightly trickier. If an observer at a point P see the Universe as isotropic, another observer at P moving relative to the first will not see isotropy.

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ADDITIONAL NOTES

We assume isotropy "along the t lines:" for any point $P \in \mathcal{M}$, if V^a is tangent to a t line, and S_1^a and S_2^a any two spacelike vectors orthogonal to V^a, there's an isometry that leaves P and V^a fixed and takes S_1^a to S_2^a.

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Under these restrictions, it follows that the metric is of the form

$$ds^2 = dt^2 - a^2(t)d\sigma^2$$

where $d\sigma^2$ is a metric on a 3d manifold of constant curvature, k.

There are three possibilities: $k > 0$, $k = 0$, $k < 0$. They can be scaled to $k = 1$, $k = 0$, $k = -1$.

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4. Calculating $a(t)$

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right]$$

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2 sin^2\theta \end{pmatrix}$$

63

$$g^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1-kr^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & -\frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{a^2(t)r^2 sin^2\theta} \end{pmatrix}$$

64

Christoffel symbols:

$$\Gamma^m_{ij} = \frac{1}{2}g^{mk}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right)$$

So

$$\Gamma^t_{ij} = \frac{1}{2}\left(\frac{\partial g_{it}}{\partial u^j} + \frac{\partial g_{tj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial t}\right)$$

i.e. (no sum over t),

$$\Gamma^t_{tj} = \Gamma^t_{jt} = \quad .$$

65

(5) Calculate $\Gamma^t_{r\theta}$, $\Gamma^t_{r\phi}$, $\Gamma^t_{\theta\phi}$.

(6) Calculate Γ^t_{rr}.

$$\Gamma^t_{rr} =$$

...and so forth.

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ADDITIONAL NOTES

Here they all are (non-zeroes):

$$\Gamma^t_{rr} = a\dot{a}/(1-kr^2) \qquad \Gamma^r_{rr} = kr/(1-kr^2)$$

$$\Gamma^t_{\theta\theta} = a\dot{a}r^2 \qquad \Gamma^t_{\phi\phi} = a\dot{a}r^2\, sin^2\theta$$

$$\Gamma^r_{tr} = \Gamma^\theta_{t\theta} = \Gamma^\phi_{t\phi} = \dot{a}/a$$

$$\Gamma^r_{\theta\theta} = -r(1-kr^2) \qquad \Gamma^r_{\phi\phi} = -r(1-kr^2)\, sin^2\theta$$

$$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = 1/r$$

$$\Gamma^\theta_{\phi\phi} = -\,sin\,\theta\, cos\,\theta \qquad \Gamma^\phi_{\theta\phi} = cot\,\theta$$

67

Then, we calculate the Ricci tensor:

$$R_{ij} \equiv R^m_{imj} \equiv \frac{\partial \Gamma^m_{ij}}{\partial u^m} - \frac{\partial \Gamma^m_{im}}{\partial u^j} + \Gamma^n_{ij}\Gamma^m_{nm} - \Gamma^n_{im}\Gamma^m_{nj}$$

$$R_{tt} = -3\ddot{a}/a$$

$$R_{rr} = (a\ddot{a} + 2\dot{a}^2 + 2k)/(1-kr^2)$$

$$R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k)$$

$$R_{\phi\phi} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\, sin^2\theta$$

68

and

$$R = g^{ij}R_{ij} = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$$

then set

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^4}T_{ij},$$

69 with T_{ij} for a perfect fluid.

ADDITIONAL NOTES