Arvind Borde / MTH 675, Unit 20: Cosmology

1. Review

(1) What do we do when we do GR?

(2) What is Einstein's equation?

where T_{ab} is the E-m tensor, $R = g^{ab}R_{ab}$,

1

and
$$\begin{aligned} R_{ab} = {} & \partial_e\big[\tfrac{1}{2}g^{ed}(\partial_a g_{bd}+\partial_b g_{ad}-\partial_d g_{ab})\big] \\ & - \partial_a\big[\tfrac{1}{2}g^{ed}(\partial_b g_{ed}+\partial_e g_{bd}-\partial_d g_{be})\big] \\ & - \big[\tfrac{1}{2}g^{fd}(\partial_a g_{ed}+\partial_e g_{ad}-\partial_d g_{ae})\big] \\ & \quad \times \big[\tfrac{1}{2}g^{ed}(\partial_f g_{bd}+\partial_b g_{fd}-\partial_d g_{fb})\big] \\ & + \big[\tfrac{1}{2}g^{ed}(\partial_a g_{bd}+\partial_b g_{ad}-\partial_d g_{ab})\big] \\ & \quad \times \big[\tfrac{1}{2}g^{fd}(\partial_e g_{fd}+\partial_f g_{ed}-\partial_d g_{ef})\big] \end{aligned}$$

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(3) What does it mean to "solve Einstein's eq.?"

(4) It's not possible to solve the equation without further assumptions. What might they be?

(Solving without assumptions is hard: we have a complicated system of coupled non-linear p.d.e.'s.)

3

The theory is difficult computationally, and also conceptually because you are determining the structure (or geometry) of spacetime itself ________.

What does knowing the metric, g_{ab}, do for you?

4

Knowing the metric allows you to:

- ○ ________
  - · ________
  - · ________
- ○ ________

5

Another twist:

Solving Einstein's equation

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab}$$

means simultaneously figuring out the metric, g_{ab},

6

ADDITIONAL NOTES

The vacuum: $T_{ab} = 0$

Einstein's equation becomes

Solutions of this are called ______________.

An obvious vacuum solution is that g_{ab} is constant. making R_{ab} and R zero.

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This means that the metric can be reduced everywhere to the Minkowski metric

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

i.e, spacetime is flat.

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That's what you'd *naively* expect:

No matter

$\downarrow$

No gravitation

$\downarrow$

No curvature

Spacetime should be flat. Life is good (at least for now).

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3. Cosmology

"*we are not able to make cosmological models without some admixture of ideology*"

– S.W.Hawking and G.F.R.Ellis
"The Large Scale Structure of Space-Time," 1973.

The ideology that we adhere to today is the anti-ideology of the anthropocentric view held in Western Europe for around 2,000 years: ______________

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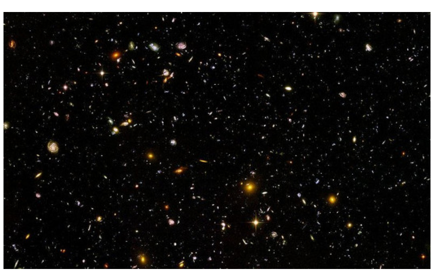
Less dramatically, we assume that the Universe is ____________________ (all places are the same) and ________________ (all directions the same) on the largest scales.



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Based *purely* on these symmetry assumptions, it's possible to pick suggestive coordinates (t, r, θ, ϕ) and fix a form for the metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \qquad \begin{cases} k = -1 \\ k = 0 \\ k = +1 \end{cases}$$

where $d\Omega^2 = d\theta^2 + sin^2\,\theta d\phi^2$.

Cred: Friedman (Soviet Union), Lemaitre (France), Robertson and Walker (USA) in the 1920s–1940s.

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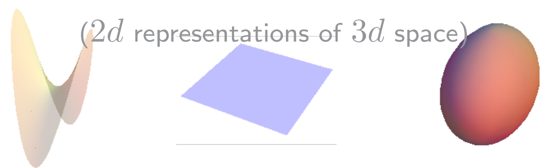
ADDITIONAL NOTES

The three possibilities, labeled by k, arise from the ________________ being

________ ________ ________

________ ________ ________



"open" universes (spatially infinite) "closed" (finite)

$0 \leqslant r < \infty$ $0 \leqslant r < 2\pi$

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$a(t)$ is called the ________________ of the Universe and is assumed positive. Its behavior follows from Einstein's equation and the nature of matter. Different behaviors arise in different circumstances.

The nature of matter is what the energy-momentum says it is. Our symmetry assumptions force matter to be uniformly sprinkled throughout the Universe.

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We model this by assuming the Universe is filled entirely and uniformly by a fluid:

$$T_{ab} = (\rho + P)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + P g_{ab}$$

where we identify ______________________________ ____________ and __________________________.

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The metric, g_{ab}, appears on the right hand side of Einstein's equation, too.

Our goal is to figure out the behavior of $a(t)$, ρ and P from Einstein's equation.

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The steps are

1. Calculate the curvatures R_{ab} and R from the FLRW metric. The answer will involve $a(t)$, and its derivatives.
2. Set up Einstein's equation with the above form of T_{ab}:

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

3. Solve (as best you can).

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The result of steps (1) and (2) is:

($\diamondsuit$)

($\heartsuit$)

where $k = -1$, $k = 0$, and $k = +1$ gives us the three cases we've mentioned.

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ADDITIONAL NOTES

(5) What does eqn. ♢ say ρ has to be if $a(t) = \mathbf{C}$ (a constant)?

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In practice, both ρ and P are positive for normal forms of matter.

(6) In this case, what does eqn. ♡ say whether $a(t)$ can be constant?

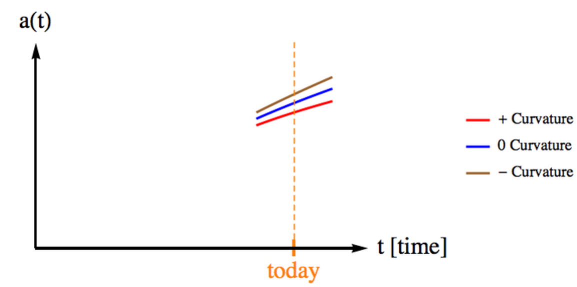
20

We can get more information by making further assumptions about ρ and P, usually a relationship between the two ("equation of state").

One scenario: $P = \rho/3$ ("radiation dominated")

$$a(t) = \begin{cases} \sqrt{2t + t^2} & (-) \\ 4^{1/4}\sqrt{t} & (0) \\ \sqrt{2t - t^2} & (+) \end{cases}$$

21 Plotting this around "today":



$\dot{a} > 0$: The Universe is expanding.

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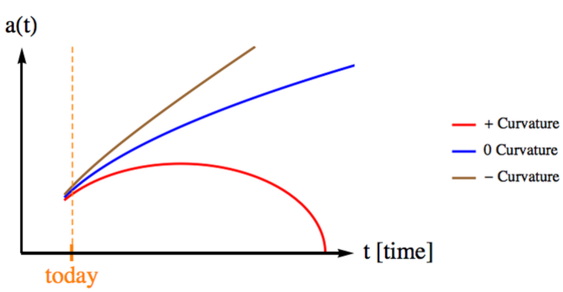
Projecting into the future:



$\ddot{a} < 0$: The expansion is deccelerating.

The graph is concave down.

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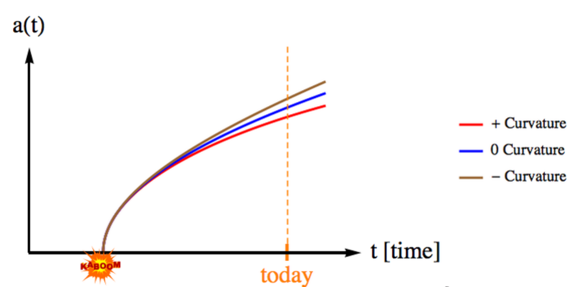
Projecting into the past:



If the graph is concave down throughout, it follows that $a(t)$ was zero in the past.

(The Universe had a beginning.)

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ADDITIONAL NOTES

This consequence holds up in a wide variety of scenarios where "gravity is attractive." This happens when $\rho > 0$ and $P > 0$, and it forces the expansion of the Universe to decelerate. I.e., the equation

$$\frac{3\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{c^4}(\rho + 3P)$$

forces $\ddot{a} < 0$. Observations indicate that the Universe *is* expanding, we are led to the inevitable conclusion: **There is always a beginning**.

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(7) Now, rewrite the ($\diamondsuit$) equation with all the $a(t)$ terms on the left.

(8) If $a(t) \to 0$ what can you say about the left-hand side, and therefore ρ on the right?

____

____

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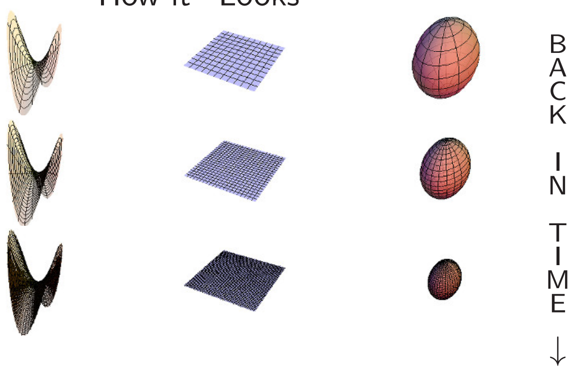
How it "Looks"



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As $t \to 0$, $a(t) \to$ ______, and $\rho \to$ __________.

____

____

____

Infinities are problematic (________________).
Still, the Universe was in a very hot, dense state in the past from which it exploded and expanded: ____________________

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You'd think Einstein would be happy at this major consequence, but he had sensed as far back as 1917 that these issues, especially that of singularities, might be a problem.

In fact, these issues arise in Newtonian gravity, as well, as Newton mentioned in letters he wrote in 1692-93 ("Four Letters from Sir Isaac Newton to Doctor Bentley").

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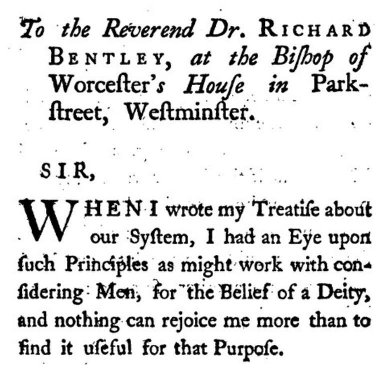

To the Reverend Dr. RICHARD BENTLEY, *at the Biſhop of* Worceſter's *Houſe in* Park-ſtreet, Weſtminſter.

SIR,

WHEN I wrote my Treatiſe about our Syſtem, I had an Eye upon ſuch Principles as might work with conſidering Men, for the Belief of a Deity, and nothing can rejoice me more than to find it uſeful for that Purpoſe.

December 10, 1692

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ADDITIONAL NOTES

As to your firſt Query, it ſeems to me that if the Matter of our Sun and Planets, and all the Matter of the Univerſe, were evenly ſcattered throughout all the Heavens, and every Particle had an innate Gravity towards all the reſt, and the whole Space, throughout which this Matter was ſcattered, was but finite; the Matter on the outſide of this Space would by its Gravity tend towards all the Matter on the inſide, and by conſequence fall down into the middle of the whole Space, and there compoſe one great ſpherical

December 10, 1692

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Maſs. But if the Matter was evenly diſpoſed throughout an infinite Space, it could never convene into one Maſs, but ſome of it would convene into one Maſs and ſome into another, ſo as to make an infinite Number of great Maſſes, ſcattered at great Diſtances from one to another throughout all that infinite Space.

December 10, 1692

Newton then goes on to say how this might explain why the Universe contains what it does, *as known at the time* (stars and planets) . . .

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thus might the Sun and fixt Stars be formed, ſuppoſing the Matter were of a lucid Nature. But how the Matter ſhould divide itſelf into two ſorts, and that Part of it, which is fit to compoſe a ſhining Body, ſhould fall down into one Maſs and make a Sun, and the reſt, which is fit to compoſe an opaque Body, ſhould coaleſce, not into one great Body, like the ſhining Matter, but into many little ones .... I do not think explicable by meer natural Cauſes,

December 10, 1692

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Bentley raised objections and Newton replied:

I Agree with you, that if Matter evenly diffuſed through a finite Space, not ſpherical, ſhould fall into a ſolid Maſs, this Maſs would affect the Figure of the whole Space, provided it were not ſoft, ... the Protuberances might ſometimes ſink a little by their Weight, and thereby the Maſs might, by Degrees, approach a ſpherical Figure.

January 17, 1693

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harder it is to ſuppoſe all the Particles in an infinite Space ſhould be ſo accurately poiſed one among another, as to ſtand ſtill in a perfect Equilibrium. For I reckon this as hard as to make not one Needle only, but an infinite number of them (ſo many as there are Particles in an infinite Space) ſtand accurately poiſed upon their Points.

January 17, 1693

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Yet I grant it poſſible, at leaſt by a divine Power; and if they were once to be placed, I agree with you that they would continue in that Poſture without Motion for ever, unleſs put into new Motion by the ſame Power. When therefore I ſaid, that Matter evenly ſpread through all Space, would convene by its Gravity into one or more great Maſſes, I underſtand it of Matter not reſting in an accurate Poiſe.

January 17, 1693

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ADDITIONAL NOTES

But you argue, in the next Paragraph of your Letter, that every Particle of Matter in an infinite Space, has an infinite Quantity of Matter on all Sides, and by confequence an infinite Attraction every way, and therefore muft reft in Equilibrio, becaufe all Infinites are equal. Yet you fufpect a Paralogifm in this Argument; and I conceive the Paralogifm lies in the Pofition, that all Infinites are equal.

January 17, 1693

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After a long discussion of infinity, Newton ends:

I fear what I have faid of Infinites, will feem obfcure to you; but it is enough if you underftand, that Infinites when confidered abfolutely without any Reftriction or Limitation, are neither equal nor unequal, nor have any certain Proportion one to another, and therefore the Principle that all Infinites are equal, is a precarious one.

January 17, 1693

Newton ties himself into knots, with good reason.

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The same issue arises in General Relativity. A year after his fundamental paper introducing that theory, Einstein wrote a paper on cosmology:

COSMOLOGICAL CONSIDERATIONS ON THE GENERAL THEORY OF RELATIVITY

BY

A. EINSTEIN

Translated from "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie," Sitzungsberichte der Preussischen Akad. d. Wissenschaften, 1917.

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He saw that difficulties with the Newtonian theory were serious

If we apply Boltzmann's law of distribution for gas molecules to the stars, by comparing the stellar system with a gas in thermal equilibrium, we find that the Newtonian stellar system cannot exist at all. For there is a finite ratio

and that General Relativity would have the same problem. A static Universe would be unstable.

40

Gravity is a destabilizing phenomenon because it is (apparently) ____________________

Newton and Einstein realized that the attractive nature of gravity made systems that are controlled by ____________ inherently prone to collapsing on themselves. Yet, much of the Universe is stable. We don't see things rushing to each other, overwhelmed by a fatal gravitational attraction.

41

Against gravitational collapse, what supports

(9) the atomic nucleus? ____________________

(10) molecules? ____________________

(11) your nose? ____________________

42

ADDITIONAL NOTES

But, what supports and gives stability to

(12) the solar system? ________________

(13) galaxies? ________________

No electromagnetic forces at work here.

43

The motion that gives stability to the solar system and to galaxies is ________________

There is no evidence for large-scale rotation of the Universe. But expansion could have been used by Einstein to stabilize the Universe.

44

Instead, he changed his theory

§ 2. The Boundary Conditions According to the General Theory of Relativity

In the present paragraph I shall conduct the reader over the road that I have myself travelled, rather a rough and winding road, because otherwise I cannot hope that he will take much interest in the result at the end of the journey. The conclusion I shall arrive at is that the field equations of gravitation which I have championed hitherto still need a slight modification, so that on the basis of the general theory of relativity those fundamental difficulties may be avoided which have been set forth in § 1 as confronting the Newtonian theory. This modification corresponds perfectly to the transi-

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admitting that

will not here be discussed. In order to arrive at this consistent view, we admittedly had to introduce an extension of the field equations of gravitation which is not justified by our actual knowledge of gravitation. It is to be emphasized,

This is a glimpse of how some of the remainder of Einstein's career would play itself out.

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Einstein proposed altering his equation to

$$R_{ab} - \frac{1}{2} R g_{ab} + \boxed{\Lambda g_{ab}} = \frac{8\pi G}{c^4} T_{ab}$$

Λ is called the ________________________________ and is meant to give a small effective large scale repulsion that would balance the instability caused by the inherently attractive nature of gravity, and so give a static Universe.

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Einstein found the idea of a static Universe attractive because, among other reasons, it would eliminate the question of the initial singularity (the origin of the Universe).

The Universe would not begin, end, or change.

It would simply be.

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ADDITIONAL NOTES

The cosmological solutions now become

$$\frac{3\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{c^4}\rho - \frac{3k}{a^2(t)} \qquad (\diamondsuit)$$

$$\frac{3\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{c^4}(\rho + 3P) + \Lambda \qquad (\heartsuit^\Lambda)$$

Einstein showed that if $k = 1$ and $P = 0$ there is a particular ("critical") value of Λ that allows a static solution with $a(t) =$ constant.

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We've seen that if $a(t) = \mathbf{C} =$ constant,

$$\rho = \frac{3kc^4}{8\pi G\mathbf{C}^2}$$

Solving this for $a(t) = \mathbf{C}$:

$$\mathbf{C} = \sqrt{\frac{3kc^4}{8\pi G\rho}}$$

50

(14) Einstein had assumed that $k = 1$. Did he need that assumption? ____________

51

(15) Use $(\heartsuit^\Lambda)$ to get the critical value of Λ, in terms of ρ, that gives Einstein's static Universe.

52

Therefore, we can write

$$\frac{3\ddot{a}(t)}{a(t)} = -\Lambda_{\text{crit}} + \Lambda$$

(16) What would happen if Λ were slightly bigger than the critical value and what would happen if it were slightly smaller?

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If $\Lambda > \Lambda_{\text{crit}}$, we have $\ddot{a}(t) > 0$. If the Universe is expanding it accelerates; if it is contracting it decelerates. The effective repulsion that Λ gives dominates the attractive nature of gravity.

If $\Lambda < \Lambda_{\text{crit}}$, we have $\ddot{a}(t) < 0$. If the Universe is expanding it decelerates; if it is contracting it accelerates. The effective repulsion that Λ gives is dominated by the attractive nature of gravity.

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ADDITIONAL NOTES

We have a single value of Λ that gives a static Universe, with very different behavior on either side.

This was criticized right away as too delicate a “fine tuning.” Further, as the years went on, it was observed that the Universe is, in fact, expanding.

Eventually Einstein came around, dropped the cosmological constant, accepted a dynamic Universe, and agreed that the Universe had a beginning.

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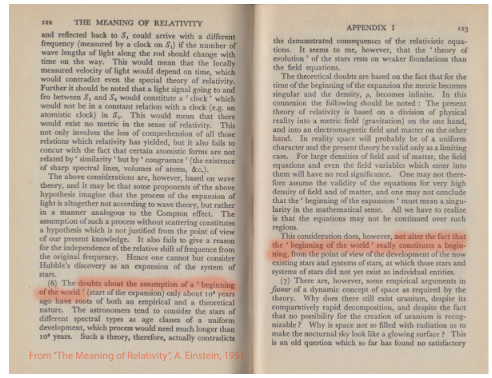

122 THE MEANING OF RELATIVITY

and reflected back to S_1 could arrive with a different frequency (measured by a clock on S_1) if the number of wave lengths of light along the rod should change with time on the way. This would mean that the locally measured velocity of light would depend on time, which would contradict even the special theory of relativity. Further it should be noted that a light signal going to and fro between S_1 and S_2 would constitute a ‘ clock ’ which would not be in a constant relation with a clock (e.g. an atomistic clock) in S_1. This would mean that there would exist no metric in the sense of relativity. This not only involves the loss of comprehension of all those relations which relativity has yielded, but it also fails to concur with the fact that certain atomistic forms are not related by ‘ similarity ’ but by ‘ congruence ’ (the existence of sharp spectral lines, volumes of atoms, &c.).

The above considerations are, however, based on wave theory, and it may be that some proponents of the above hypothesis imagine that the process of the expansion of light is altogether not according to wave theory, but rather in a manner analogous to the Compton effect. The assumption of such a process without scattering constitutes a hypothesis which is not justified from the point of view of our present knowledge. It also fails to give a reason for the independence of the relative shift of frequence from the original frequency. Hence one cannot but consider Hubble's discovery as an expansion of the system of stars.

(6) The doubts about the assumption of a ‘ beginning of the world ’ (start of the expansion) only about 10^9 years ago have roots of both an empirical and a theoretical nature. The astronomers tend to consider the stars of different spectral types as age classes of a uniform development, which process would need much longer than 10^9 years. Such a theory, therefore, actually contradicts

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the demonstrated consequences of the relativistic equations. It seems to me, however, that the ‘ theory of evolution ’ of the stars rests on weaker foundations than the field equations.

The theoretical doubts are based on the fact that for the time of the beginning of the expansion the metric becomes singular and the density, ρ, becomes infinite. In this connexion the following should be noted: The present theory of relativity is based on a division of physical reality into a metric field (gravitation) on the one hand, and into an electromagnetic field and matter on the other hand. In reality space will probably be of a uniform character and the present theory be valid only as a limiting case. For large densities of field and of matter, the field equations and even the field variables which enter into them will have no real significance. One may not therefore assume the validity of the equations for very high density of field and of matter, and one may not conclude that the ‘ beginning of the expansion ’ must mean a singularity in the mathematical sense. All we have to realize is that the equations may not be continued over such regions.

This consideration does, however, not alter the fact that the ‘ beginning of the world ’ really constitutes a beginning, from the point of view of the development of the now existing stars and systems of stars, at which those stars and systems of stars did not yet exist as individual entities.

(7) There are, however, some empirical arguments in *favour* of a dynamic concept of space as required by the theory. Why does there still exist uranium, despite its comparatively rapid decomposition, and despite the fact that no possibility for the creation of uranium is recognizable? Why is space not so filled with radiation as to make the nocturnal sky look like a glowing surface? This is an old question which so far has found no satisfactory

From “The Meaning of Relativity”, A. Einstein, 1951

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ADDITIONAL NOTES