Arvind Borde / MTH 675, Unit 2: Vectors

1. General Vector Spaces

A vector space over a field F is a set V on which two operations, "vector addition" (denoted by $+$ and "scalar multiplication" (usually denoted by juxtaposition), are defined. For example:

Vector addition: $\vec{\boldsymbol{U}} + \vec{\boldsymbol{V}}$

Scalar multiplication: $x\vec{\boldsymbol{V}}$

where $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{V}}$ denote vectors, and $x \in F$ is a "scalar" (it scales the vector that it multiplies).

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Vector addition must satisfy the following:

- Closure: If $\vec{\boldsymbol{U}}$, $\vec{\boldsymbol{V}} \in V$, then $\vec{\boldsymbol{U}} + \vec{\boldsymbol{V}} \in V$.
- Commutative law: $\forall\, \vec{\boldsymbol{U}}, \vec{\boldsymbol{V}} \in V$,
$$\vec{\boldsymbol{U}} + \vec{\boldsymbol{V}} = \vec{\boldsymbol{V}} + \vec{\boldsymbol{U}}.$$
- Associative law A: $\forall\, \vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}, \vec{\boldsymbol{W}} \in V$,
$$\vec{\boldsymbol{U}} + (\vec{\boldsymbol{V}} + \vec{\boldsymbol{W}}) = (\vec{\boldsymbol{U}} + \vec{\boldsymbol{V}}) + \vec{\boldsymbol{W}}.$$

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- Additive identity: The set V contains an "additive identity element," denoted by $\vec{\boldsymbol{0}}$, such that for any $\vec{\boldsymbol{V}} \in V$, $\vec{\boldsymbol{0}} + \vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{V}} + \vec{\boldsymbol{0}} = \vec{\boldsymbol{V}}$.
- Additive inverses: For each $\vec{\boldsymbol{V}} \in V$, there's an element, its additive inverse, denoted by $-\vec{\boldsymbol{V}}$, such that $\vec{\boldsymbol{V}} + (-\vec{\boldsymbol{V}}) = \vec{\boldsymbol{0}}$ and $(-\vec{\boldsymbol{V}}) + \vec{\boldsymbol{V}} = \vec{\boldsymbol{0}}$.

Note: Vector subtraction is defined through
$$\vec{\boldsymbol{U}} - \vec{\boldsymbol{V}} \equiv \vec{\boldsymbol{U}} + (-\vec{\boldsymbol{V}}).$$

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Scalar multiplication must satisfy the following:

- Closure: $\forall\, \vec{\boldsymbol{V}} \in V$, $x \in F$, $x\vec{\boldsymbol{V}} \in V$.
- Distributive law A: $\forall\, x \in F$ and all $\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}} \in V$
$$x(\vec{\boldsymbol{U}} + \vec{\boldsymbol{V}}) = x\vec{\boldsymbol{U}} + x\vec{\boldsymbol{V}}.$$
- Distributive law B: $\forall\, x, y \in F$ and all $\vec{\boldsymbol{V}} \in V$,
$$(x + y)\vec{\boldsymbol{V}} = x\vec{\boldsymbol{V}} + y\vec{\boldsymbol{V}}.$$

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- Associative law B: $\forall\, x, y \in F$, $\vec{\boldsymbol{V}} \in V$,
$$x(y\vec{\boldsymbol{V}}) = (xy)\vec{\boldsymbol{V}}.$$
- Unitary law: $\forall\, \vec{\boldsymbol{V}} \in V$, $1\vec{\boldsymbol{V}} = \vec{\boldsymbol{V}}$ (where 1 is the multiplicative identity of the field).

Note: We'll sometimes write expressions such as $\frac{\vec{\boldsymbol{V}}}{x}$. By that we'll mean $\frac{1}{x}\vec{\boldsymbol{V}}$, where $\frac{1}{x}$ is the multiplicative inverse of x in the field F.

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2. An Important Example

Vectors here are n-tuples of real numbers:

and scalars are individual real numbers.

Vector addition and scalar multiplication are defined as follows:

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ADDITIONAL NOTES

Vector addition: For any two elements of $\mathbb{R}^n$, $\vec{\boldsymbol{U}} = (u_1, u_2, \ldots, u_n)$ and $\vec{\boldsymbol{V}} = (v_1, v_2, \ldots, v_n)$,

$$\vec{\boldsymbol{U}} + \vec{\boldsymbol{V}} =$$

Scalar multiplication: For any scalar x (i.e., any real number) and any vector $\vec{\boldsymbol{V}} = (v_1, v_2, \ldots, v_n)$,

$$x\vec{\boldsymbol{V}} =$$

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(1) Can we check that the requirements for a vector space are met in this example?

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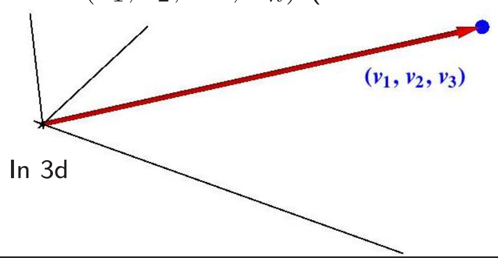
There's a nice geometrical representation of this example. The vector $\vec{\boldsymbol{V}} = (v_1, v_2, \ldots, v_n)$ can be shown as an arrow from the origin to the point with coordinates $(v_1, v_2, \ldots, v_n)$ (hence our notation).



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We'll stay with this example, unless otherwise stated.

For a vector $\vec{\boldsymbol{V}} = (v_1, v_2, \ldots, v_n)$ we'll refer to $(v_1, v_2, \ldots, v_n)$ as its ____________________

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3. Norm and Dot Product

The length ("norm") of a vector is given by

$$\|\vec{\boldsymbol{V}}\| =$$

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The norm is related to a more general operation that can be performed on vectors in $\mathbb{R}^n$, variously called the inner product, or the dot product:

$$\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{V}} \equiv \langle \vec{\boldsymbol{U}}, \vec{\boldsymbol{V}} \rangle =$$

This product has the following properties:

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ADDITIONAL NOTES

1) $\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{V}} = \vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{U}}$.

2) $(x\vec{\boldsymbol{U}} + y\vec{\boldsymbol{V}}) \cdot \vec{\boldsymbol{W}} = x\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{W}} + y\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{W}}$.

3) $\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}} > 0 \quad \forall \vec{\boldsymbol{V}} \neq \vec{\boldsymbol{0}}$.

(2) How are the norm and the dot product related?

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(3) What is the norm of $\vec{\boldsymbol{U}} = \dfrac{\vec{\boldsymbol{V}}}{\|\vec{\boldsymbol{V}}\|}$?

$$\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{U}} =$$

We call vectors ________________________

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Two vectors $\vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{W}}$ ________________ In that plane, the angle, θ, between the two is

To see this, we use the law of cosines.

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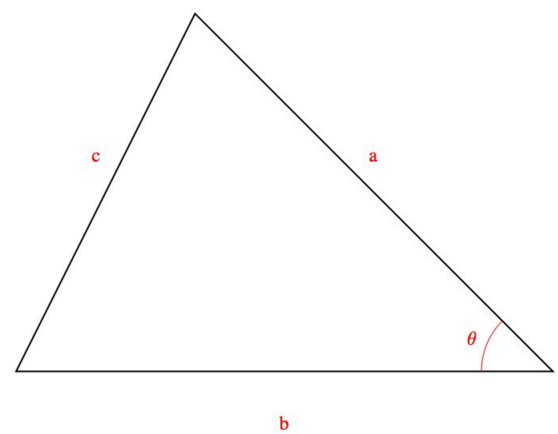
Law of cosines



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Then

$$c^2 =$$

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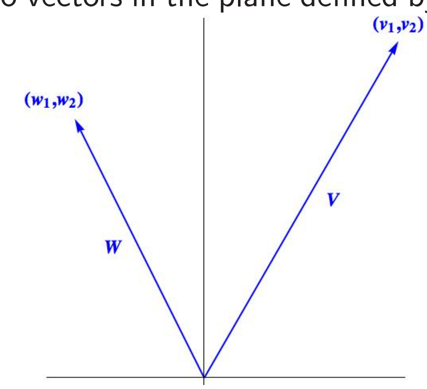
Consider two vectors in the plane defined by them:



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ADDITIONAL NOTES

You can get it two ways:

1) ______________________________

and

2) ____________________

1) $\ell^2 =$

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2) $\ell^2 =$

Equating the previous answer to this and canceling like crazy, you get

(4) Does the lhs look familiar? __________

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So

or

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4. Orthogonality of Vectors

Orthogonality tries to capture the idea of perpendicularity.

(5) If two vectors, $\vec{W}$ and $\vec{V}$, are perpendicular, what is, θ, the angle between them? ____

(6) What is $cos\,\theta$? __

(7) Therefore, what is $\vec{W} \cdot \vec{V}$? __

We call $\vec{W}$ and $\vec{V}$ ________ iff $\vec{W} \cdot \vec{V} = 0$.

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5. Vector/Cross Product

The cross product, $\vec{V} \times \vec{W}$, of two 3d vectors $\vec{V}$ and $\vec{W}$ is defined as the vector with components given by

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Properties of the cross product:

- $\vec{V} \times \vec{W} = -(\vec{W} \times \vec{V})$.
- $\vec{V} \times \vec{V} = \vec{0} \quad \forall \vec{V}$.
- $\vec{U} \times (\vec{V} + \vec{W}) = \vec{U} \times \vec{V} + \vec{U} \times \vec{W}$.
- $(x\vec{V}) \times (y\vec{W}) = xy(\vec{V} \times \vec{W})$.
- $\vec{U} \cdot (\vec{V} \times \vec{W}) = \vec{V} \cdot (\vec{W} \times \vec{U}) = \vec{W} \cdot (\vec{U} \times \vec{V})$.

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ADDITIONAL NOTES

(8) What is $\vec{V} \cdot (\vec{V} \times \vec{W})$? ___

(9) What is $\vec{W} \cdot (\vec{V} \times \vec{W})$? ___

(10) Why?

____

____

____

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6. Vector-Valued Functions of Real Variables

If t is a real variable, a vector-valued function of t, $\vec{V}(t)$, is defined as

$$\vec{V}(t) =$$

We'll assume each of the $v_i(t)$ are *differentiable* real-valued functions of t.

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This allows us to define the derivative of $\vec{V}(t)$ with respect to t as

$$\frac{d\vec{V}}{dt} =$$

From the various rules of differentiation for real-valued functions, matching rules for vector-valued functions follow:

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- $\frac{d}{dt}(\vec{V} + \vec{W}) =$
- $\frac{d}{dt}\left(f(t)\vec{V}\right) =$
- $\frac{d}{dt}\left(\frac{1}{f(t)}\vec{V}\right) =$

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- $\frac{d}{dt}(\vec{V} \cdot \vec{W}) =$
- $\frac{d}{dt}(\vec{V} \times \vec{W}) =$

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Changing the independent variable:

Suppose $\vec{V}(t)$ is a vector-valued function of the variable t. Suppose s is a (differentiable) real-valued function of t. Then the chain-rule for real-valued functions leads to

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ADDITIONAL NOTES

And, if $\vec{V}$ is a vector-valued function of two variables, say u and v, that are themselves functions of a variable s, then

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7. "Unusual" Vector Spaces

Our $\mathbb{R}^n$ example inspired the concept of an abstract vector space.

But, what does abstraction give us? Are there other types of vectors out there besides arrows?

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1. Solutions of certain differential equations obey the rules that make them a vector space.
2. A cuter example is this:

$$\left\langle {a \atop b} \right\rangle$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$.

That's all very well, but how do we add the darn things, and what's scalar multiplication?

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- Vector addition: $\left\langle {a \atop b} \right\rangle \oplus \left\langle {c \atop d} \right\rangle = \left\langle {a+c \atop b \cdot d} \right\rangle$

Example:

$$\left\langle {2 \atop 3} \right\rangle \oplus \left\langle {1 \atop 4} \right\rangle =$$

- Scalar multiplication: $x\left\langle {a \atop b} \right\rangle = \left\langle {ax \atop b^x} \right\rangle$

Example:

$$5\left\langle {2 \atop 3} \right\rangle =$$

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(11) What's the identity under this addition?

(12) What's the additive inverse of $\left\langle {a \atop b} \right\rangle$

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ADDITIONAL NOTES