Definition [Atlas] An n-dimensional smooth atlas of a Hausdorff topological space, $\mathcal{M}$, is a collection of open sets, O_i, each with a chart, σ_i, such that

- $\bigcup O_i = \mathcal{M}$, and
- for every i, j, $\sigma_j^{-1}\sigma_i$ is a smooth mapping of $\sigma_i^{-1}(O_i \cap O_j) \subset \mathbb{R}^n$ to $\mathbb{R}^n$.

The second says that charts are smooth functions of each other in overlap regions.

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Definition [Manifold] An n-dimensional manifold, $\mathcal{M}$, is a Hausdorff topological space equipped with an n-dimensional smooth atlas.

This is a barebones entity, so far, In order to be able to talk about curvature, etc., we'll need additional structure.

The essential additional structure we need is the metric.

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1. Metrics

Let $\mathcal{M}$ be a manifold and let V_p be the tangent space at $P \in \mathcal{M}$ (the vector space of tangent vectors at P).

A metric, g, on $\mathcal{M}$ is a symmetric, nondegenerate linear map from $V_p \times V_p \to \mathbb{R}$.

Symmetric: $g(\vec{V}, \vec{W}) = g(\vec{W}, \vec{V})$.

3 Nondegenerate: $g(\vec{V}, \vec{W}) = 0 \, \forall \vec{W} \iff \vec{V} = \vec{0}$.

We do not require that the metric be positive definite, i.e., that $g(\vec{V}, \vec{V}) > 0 \, \forall \vec{V} \neq \vec{0}$.

Given a basis $\{\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_n\}$ of V_p, we define

$$g_{ij} \equiv g(\vec{V}_i, \vec{V}_j)$$

It's always possible to choose an orthonormal basis in which

4 $$g_{ij} = \pm\delta_{ij}.$$

The number of $+1$s and -1s is independent of choice of orthonormal basis.

We call this pair of numbers the signature of the metric.

A metric with signature $(1, n-1)$ is called an nd Lorentz metric, and $\mathcal{M}$ is called a spacetime.

The single $+1$ distinguishes "time" from "space."

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For example a 4d Lorentz metric is a metric that can be reduced by an appropriate choice to the diagonal form

$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

i.e., to the 4d Minkowski metric, η_{ij}.

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ADDITIONAL NOTES

Just as in Minkowski spacetime, we can classify vectors, V^a, in a general Lorentz spacetime into three types:

$$\text{If } g_{ab}V^aV^b \text{ is } \begin{cases} > 0, & V^a \text{ is timelike.} \\ = 0, & V^a \text{ is null.} \\ < 0, & V^a \text{ is spacelike.} \end{cases}$$

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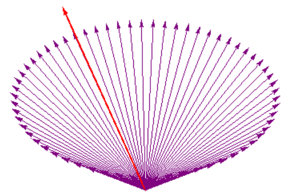
The local future light cone at any point is the set of future-directed null vectors there:



There are two lightcones at every point. We assume we can consistently pick one as "future" and one as "past."

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Assume that curves are smooth.

We define a

- timelike curve: tangent timelike at every point.
- null curve: tangent null at every point.
- spacelike curve: tangent spacelike at every point.

Timelike curves are assumed to be more than a single point ("non degenerate").

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We define the causal relation "$\ll$" by

$p \ll q \Leftrightarrow$ there is a future-directed timelike curve from p to q.

The event p is considered to occur before q and to be able to send q a signal. We sometimes assume that $p \not\ll p$ for any p.

(1) What does this say in words?

p cannot occur before p – there is no "time travel."

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It follows that

if $p \ll q$ and $q \ll r$, then $p \ll r$.

(2) What does this say in words?

If p occurs before q, and q occurs before r, then p occurs before r.

Or, if p can send a signal to q, and q can send a signal to r, then p can send a signal to r.

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We define

$$I^+(p) = \{q \mid p \ll q\}$$

as the chronological future of p, and

$$I^-(p) = \{q \mid q \ll p\}$$

as its chronological past.

These definitions can be extended to sets of points.

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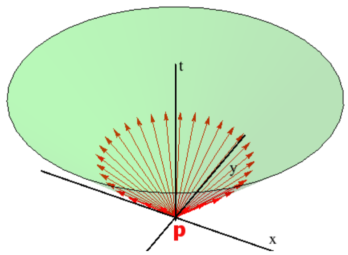
ADDITIONAL NOTES

What these definitions are trying to get at is a generalization of the idea of a Minkowski light cone:

The arrows are the future-pointing null vectors at some point and the shaded cone is the straight-line continuation of these vectors.



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In a general spacetime the full light cone is a straightline extension of the local one, and the interior of that cone is the set of all points that p can communicate with via timelike lines.

In curved spacetime, the local future lightcone of p is still the set of future-pointing null vectors there, but the global light cone can be quite complicated and is harder to define.

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Therefore, we first define $I^{\pm}(p)$.

The interior of the full future light cone in a curved spacetime is $I^{+}(p)$, the chronological future of p. It is the set of all points that p can communicate with via timelike curves.

The "global future lightcone" of a point is replaced by $\dot{I}^{\pm}(p)$, the boundary of $I^{\pm}(p)$.

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2. The Principle of Equivalence

The reduction of g_{ij}, in an appropriate basis, to η_{ij} fulfills an important physical requirement: Just as acceleration can mimic gravitation, an appropriate reference frame can "cancel gravity locally."

That either happens – "cancellation" or "creation" of gravity(like) effects – is because of the principle of equivalence.

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In Newtonian physics: forces cause accelerations.

How much acceleration?
Given by Newton's second law of motion, $F = ma$, where m represents the mass, a measure of the resistance to the force, the inertia. For the same force, the greater the mass, the smaller the acceleration.

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Forces are given by "force laws."

Example 1: Electrostatic Forces:

Coulomb's Law

$$F_{\text{elec}} = k\frac{q_1 q_2}{d^2},$$

$$k \approx 9 \times 10^9 N \cdot m^2/C^2$$

Coulomb constant

(Charge measured in Coulombs.)

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ADDITIONAL NOTES

Coulomb's law tells you that the electrical force between two objects with charges q_1 and q_2 is attractive or repulsive and points in the direction of the straight line between them.

$$q_1(+)\bullet\underbrace{\rightarrow\text{attractive electrostatic force }\leftarrow}_{distance\,=\,d}\circ q_2(-)$$

$$q_1(+)\bullet\underbrace{\leftarrow\text{repulsive electrostatic force }\rightarrow}_{distance\,=\,d}\circ q_2(+)$$

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It says that the force on a charge q "due to" an "external charge" Q, $k\dfrac{qQ}{d^2}$, is proportional to q.

If q doubles, the force on it doubles.

We call the electrical influence of external charges, etc., the electric field.

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Now, a Helium nucleus has twice the charge of a Hydrogen nucleus and roughly four times its mass.

(3) Facing the same electric field, which of the two will experience a greater electric force and by how much?
The Helium.
It will experience twice the force, because $F = qE$ and $q_{\text{He}} = 2q_{\text{H}}$.

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(4) Which will accelerate more in this situation (and why)?
The Hydrogen nucleus.
The Helium nucleus feels twice the force, but has four times the inertia, and resists the force four times as strongly.

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Example 2: Gravitational Forces:

Newton's Law of Universal Gravitation

$$F_{\text{grav}} = G\frac{m_1 m_2}{d^2},$$

$$\underbrace{G = 6.67\times 10^{-11}}_{\text{Gravitational constant}}\frac{N\cdot m^2}{kg^2}$$

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This law tells you that the gravitational force between two objects with masses m_1 and m_2 is always attractive and points in the direction of the straight line between them.

$$m_1\bullet\underbrace{\rightarrow\text{attractive gravitational force }\leftarrow}_{distance\,=\,d}\circ m_2$$

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(5) Subject to the same gravitational field which of the two nuclei, Hydrogen or Helium, will experience a greater gravitational force and by how much?

The Helium.

It will experience four times the force, because $m_{\text{He}} = 4m_{\text{H}}$.

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(6) Which will accelerate more?

They will accelerate the same, because the Helium nucleus, although it feels four times the force, also has four times the inertia.

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Electrostatic and grav. forces are similar, but:

In electrostatics, charge creates force, and mass (inertia) resists it.

In gravitation, mass creates force, and mass (inertia) resists it.

The principle of equivalence says that the mass that creates gravitational forces (gravitational mass) equals the mass that resists forces (inertial mass).

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3. A Model for the World

We model the world as a connected, Hausdorff manifold with a Lorentz metric, g_{ij}, defined on it.

Postulate: In the absence of other forces, small and "unmassive" enough particles ("test particles") follow geodesic paths of g_{ij}.

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From U13, a curve $\{u^i(t)\}$ is a geodesic with parameter t if its tangent $V^i(t) = du^i/dt$ obeys

$$\frac{dV^i}{dt} + \Gamma^i_{kj} V^k V^j = 0.$$

The equation will preserve this form under linear reparametrizations, but not nonlinear ones. The class of parametrizations under which the geodesic equation has this form is called "affine."

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Geodesics inherit timelike, null and spacelike characteristics from the class of all curves.

In an important theorem, Penrose proved that through every point of the boundary $\dot{I}^{\pm}(p)$ there passes a past-directed null geodesic that does not leave the boundary, except through p.

This structure generalizes that of the global light cone of p in Minkowski spacetime.

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ADDITIONAL NOTES

4. Obtaining the Metric

If we knew the metric, we could calculate geodesics and thus predict the paths that test bodies follow – rays of light, planets, etc.

Einstein set up a differential equation for obtaining g_{ij} by linking curvature (which would represent gravity) with matter ("mass"), as an extension of Newtonian theory.

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But what is "matter"?

Special relativity, and mass-energy equivalence, had necessitated extending our notions of mass and energy into a unified "energy-momentum tensor."

Example (4d): an idealized fluid characterized by an energy-density ρ, a pressure P, and a "flow" vector field V^i. In the rest frame of the fluid,

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$$V^i = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$T_{ij} = (\rho + P) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + P\eta_{ij}$$

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The rule of thumb that relativists follow, following Einstein, is to take special-relativistic expressions that have η_{ij} in them and replace η_{ij} with g_{ij}:

$$T_{ij} = (\rho + P) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + Pg_{ij}$$

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Then, if in Newtonian theory,

Gravity $\sim$ Mass,

a new, extended, geometrical theory should have

Spacetime curvature $\sim$ E-m tensor.

A more specific suggestion came from Poisson's equation for the gravitational field of a continuous mass distribution of mass-density ρ:

$$\nabla^2\phi = 4\pi\rho.$$

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The quantity ϕ here is the gravitational potential, the gravitational potential energy per unit mass. It was known that the first derivative of ϕ gave the gravitational force, and the second derivative gave the differential, or "tidal" force.

A study of geodesics in curved geometry suggested that it was the curvature that represented "tidal" effects in geodesic behavior.

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ADDITIONAL NOTES

By the time Einstein was working through these ideas, it was well-established that the Reimann-Christoffel tensor represented curvature.

But it couldn't simply be equated to the energy-momentum tensor.

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A tensor introduced by Ricci, the Ricci curvature,

$$R_{ij} \equiv R^k ikj \equiv \sum_k R^k ikj$$

came to the rescue. Einstein initially tried to set the Ricci curvature proportional to T_{ij}, but it turned out to violate mass-energy/momentum conservation.

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He realized in 1915 that the following had, at least, no obvious problems:

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij}$$

We call this Einstein's equation, and G_{ij} the Einstein tensor.

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This looks a bit like Poisson's equation:

$$\nabla^2 \phi = 4\pi\rho.$$

But there's a significant difference, and not just in complexity.

In Poisson's equation, the r.h.s. is assumed known, and we have to find ϕ.

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In Einstein's equation, g_{ij} is the unknown, but it appears *on both the left- and right-hand sides*, as (for example) in the case of T_{ij} for a perfect fluid.

Conceptually a matter "distribution" is meaningless until we know what "where" is and "when" is, and we only know that *after* we know g_{ij}.

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5. Summary

g_{ij}	$\sim$	gravitational potential
Γ^k_{ij}	$\sim$	gravitational force
R^m_{ijk}	$\sim$	tidal gravitational force

Einstein's Equation (with $G = 1$ and $c = 1$)

$$G_{ij} = 8\pi T_{ij}$$

determines the metric. What could be simpler?

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ADDITIONAL NOTES

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