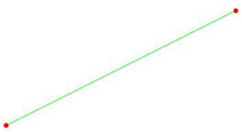
Arvind Borde / MTH 675, Unit 14: Geodesics

1. The Idea of a Geodesic

The idea that a ______________ tries to generalize is that of a straight line in flat space.



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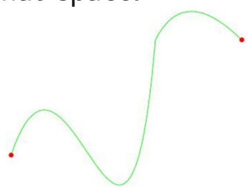
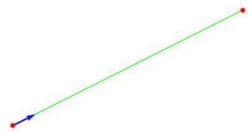


(1) What's a straight line?

1 ____

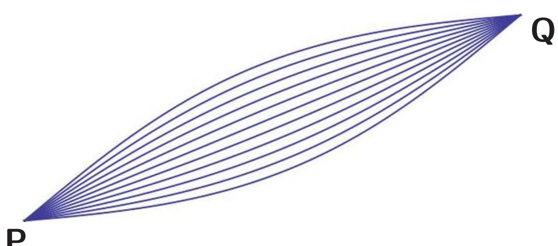
2 A straight line in flat space has the property that it's not only "the shortest" kid on the block, it's also _______________ kid on the block.
How might we define "straight" geometrically?



NOT

3 So, we face a choice: is it "shortness" or "straightness" that makes a curve, $\alpha(t)$, between points **P** and **Q** a geodesic?

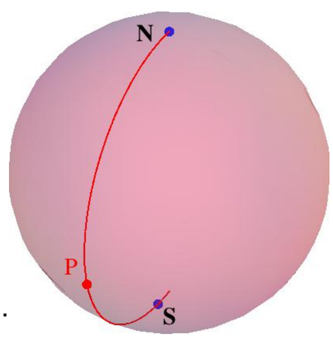
4 We have to set up a parametrized family of curves, between two points **P** and **Q** and minimize the length with respect to this parameter:



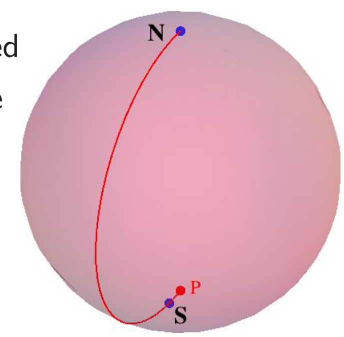
This requires "the calculus of variations."

5 There's a further difficulty with viewing geodesics as lines of extremal distance. Consider this:

If the line shown from **N** to **S** is an arc of a great circle, then the arc from **N** to **P** is the line of shortest distance between the points, hence geodesic.

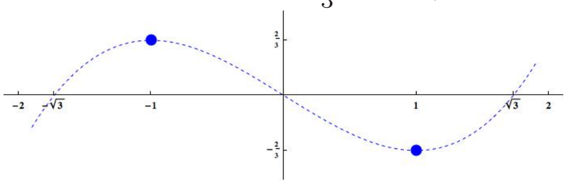


6 But what if **P** were moved as shown? On the one hand it's on an extension of the same arc of the same great circle.



ADDITIONAL NOTES

This may not seem so disconcerting. That there might a wholly different, distant extremum is something we're familiar with. Calculus is good at sniffing out local stuff, not global. The marked points below are local extrema of $\frac{1}{3}x^3 - x$,



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Still, the marked points on the previous graph *are* local extrema.

Can a geodesic be defined as a curve of *locally* minimum length? (Meaning compared to nearby curves.)

Here, too, there are difficulties.

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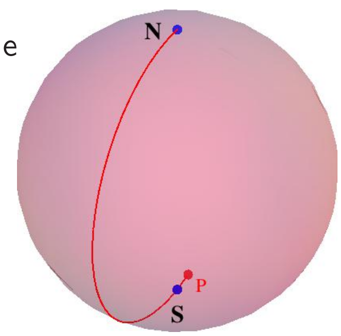
Ignoring curves "down the backside" can we regard the arc of the great circle between **N** and **P** as the locally shortest path?



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None of these difficulties is insurmountable, but they do mean that defining geodesics on the idea of "shortest distance" will involve subtleties.

That's why we'll base our approach on "straightness" or "straightestpossibleness."

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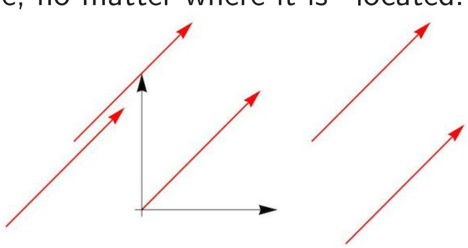
2. Parallelism

We've used the idea that a vector in $\vec{V}$ in $\mathbb{R}^n$ is "the same" if its components $(V_1, V_2, \ldots, V_n)$ are the same, no matter where it is "located."



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We're implicitly saying that vectors are "the same" if they're transported to another location, but kept __________ to their original selves.

This is OK, because of the isomorphism of the tangent spaces at each point in $\mathbb{R}^n$.

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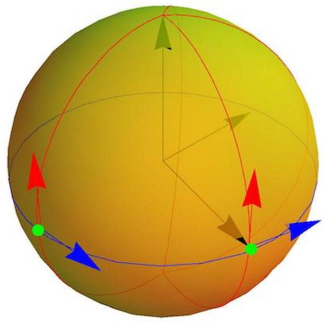
ADDITIONAL NOTES

But, what of tangent vectors at different points on a curved entity?

The vertical vectors here are clearly "the same," but what of the horizontal? These point in different directions *in the embedding space*, but would appear to point in the same direction *on the sphere*.



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We need a definition of "staying the same."

Now, broadly speaking, when we say that some quantity "stays the same" we mean that it does not ________; or being calcusians, we require that its ____________ vanish.

Easier said than done.

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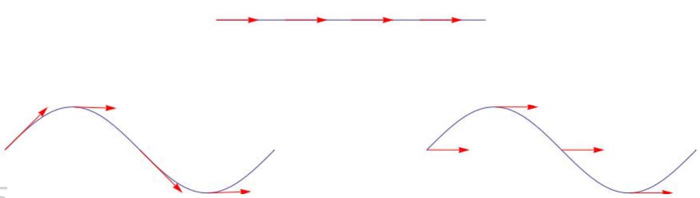
We have to distinguish between changes that occur because the *surface on which these quantities are defined* is itself curved, and changes *with respect to this surface*.



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3. The Geodesic Equation

Let $\boldsymbol{\alpha}(s)$ be a curve parameterized by its arc length *on* $\mathcal{M}$. Then $\boldsymbol{\alpha}' \cdot \boldsymbol{\alpha}' =$ ___, and so $\boldsymbol{\alpha}' \cdot \boldsymbol{\alpha}'' =$ ___. We've seen (Unit 12) that

$$\boldsymbol{\alpha}'' = \boldsymbol{\alpha}''_{\text{tan}} + \boldsymbol{\alpha}''_{\text{nor}}$$

Where $\boldsymbol{\alpha}''_{\text{tan}} =$ $\qquad\qquad\qquad\qquad$ $\vec{\boldsymbol{X}}_k$, and

$\boldsymbol{\alpha}''_{\text{nor}} =$ $\qquad\qquad\qquad$ $\vec{\boldsymbol{U}}$.

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We regard the $\boldsymbol{\alpha}''_{\text{nor}}$ as a curvature reflecting the bending of the surface on which $\boldsymbol{\alpha}$ resides, and $\boldsymbol{\alpha}''_{\text{tan}}$ as the curvature of $\boldsymbol{\alpha}$ *relative to the surface*.

Now $\boldsymbol{\alpha}''_{\text{tan}} \cdot \vec{\boldsymbol{U}} =$ ___. Further

$\boldsymbol{\alpha}''_{\text{tan}} \cdot \boldsymbol{\alpha}' =$

Therefore $\boldsymbol{\alpha}''_{\text{tan}}$ is orthogonal to both $\vec{\boldsymbol{U}}$ and $\boldsymbol{\alpha}'$, which are each unit and orthogonal to each other.

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So

$\boldsymbol{\alpha}'' \propto$

(2) What is $\vec{\boldsymbol{W}} \cdot \vec{\boldsymbol{W}}$? ___

We define the ______________________ of a curve $\boldsymbol{\alpha}(s)$, parametrized by arc length, ______, by

$\boldsymbol{\alpha}''_{\text{tan}} =$

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ADDITIONAL NOTES

(3) Find k_g.

We call $\boldsymbol{\alpha}(s)$ a geodesic if $k_g = 0$, or, equivalently, $\boldsymbol{\alpha}''_{\text{tan}} = \vec{\mathbf{0}}$. In components:

(Reminder $\boldsymbol{\alpha}' = u^{i\prime} \vec{\boldsymbol{X}}_i$.)

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4. The Covariant Derivative

Suppose $\vec{\boldsymbol{V}}$ is a vector at a point $\mathbf{P} \in \mathcal{M}$. we define the ____________________ of a scalar function f on $\mathcal{M}$ in the direction $\vec{\boldsymbol{V}}$ by

and $\boldsymbol{\alpha}(t)$ is a curve with $\boldsymbol{\alpha}(0) = \mathbf{P}$ and $\boldsymbol{\alpha}'(0) = \vec{\boldsymbol{V}}$.

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With $\boldsymbol{\alpha}(0) = \mathbf{P}$ and $\boldsymbol{\alpha}'(0) = \vec{\boldsymbol{V}}$, as before, suppose $\vec{\boldsymbol{W}}$ is a vector field on $\mathcal{M}$. Then $\vec{\boldsymbol{W}}(\boldsymbol{\alpha}(t))$ will be a vector field along $\boldsymbol{\alpha}$.

The ______________________________
______ is

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We define the ____________________ of a vector field $\vec{\boldsymbol{W}}$ in the direction $\vec{\boldsymbol{V}}$ as the component of its directional derivative that is tangent to $\mathcal{M}$:

Given a curve $\boldsymbol{\alpha}$ in $\mathcal{M}$, we say that a vector field $\vec{\boldsymbol{W}}$ is covariantly constant or parallel along it if we have $\nabla_{\vec{\alpha'}(t)} \vec{\boldsymbol{W}} = 0$ for all t.

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What is the connection with geodesics?

It turns out that $\nabla_{\vec{\alpha'}(t)} \vec{\boldsymbol{V}} = 0$ is precisely the same as saying

$$u^{k\prime\prime} + \Gamma^k_{ij} u^{i\prime} u^{j\prime} = 0.$$

In other words the geodesic equations says that a tangent to a geodes parallel transported along it stays tangent to it.

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ADDITIONAL NOTES