Appendix: More on Indices

An indexed quantity is a convenient abbreviation for a collection of quantities (functions, numbers, whatever).

To be concrete, let us assume that all indices run from 1 to 3.

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Thus, Q_i stands for the collection $\{Q_1, Q_2, Q_3\}$.
Q_{ij} stands for
$\{Q_{11}, Q_{12}, Q_{13}, Q_{21}, Q_{22}, Q_{23}, Q_{31}, Q_{32}, Q_{33}\}$.
Q_{ijk} stands for
$\{Q_{111}, Q_{112}, Q_{113}, Q_{121}, Q_{122}, Q_{123}, Q_{131}, Q_{132}, Q_{133},$
$Q_{211}, Q_{212}, Q_{213}, Q_{221}, Q_{222}, Q_{223}, Q_{231}, Q_{232}, Q_{233},$
$Q_{311}, Q_{312}, Q_{313}, Q_{321}, Q_{322}, Q_{323}, Q_{331}, Q_{332}, Q_{333}\}$
and so forth.

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One can view indexed quantities as functions on subsets of $\mathbb{N}$.

For example the 3-d Kronecker delta symbol can be thought of as a function

$$\delta(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where $i, j \in \{1, 2, 3\}$, *or* as an "indexed quantity."

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Indexing is not necessary, just convenient. Gauss did perfectly well (in 2d) by calling the elements of the first fundamental form E, F and G.

It's useful, though, to think of these as components of the indexed quantity g_{ij} where
$g_{11} = E$,
$g_{12} = g_{21} = F$, and
$g_{22} = G$.

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Conventions on index placement.

Metric: g_{ij}.
Inverse of metric: g^{ij}.
Kronecker delta: δ^j_i.
Components of a vector: V^i.

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Einstein convention on summation.

repeated lower-upper pairs are summed over:

$$g^{ij} g_{jk} \equiv \sum_j g^{ij} g_{jk}$$

$$g_{ij} V^j \equiv \sum_j g^{ij} V j$$

and so forth.

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ADDITIONAL NOTES

"Raising" and "lowering" indices.

It's useful to define quantities such as

$$V_i \equiv g_{ij} V^j$$

$$\Gamma_{ijk} = \Gamma_{ij}^m g_{mk}$$

$$Q_j^i = g^{ik} Q_{kj}, \text{ etc.}$$

37 and to *call* this "lowering" / "raising" an index.

ADDITIONAL NOTES