Arvind Borde / MTH 675, Unit 13: Gauss's Remarkable Theorem

1. Intrinsic vs Extrinsic Geometry

The ________________________ of an entity refers to geometrical properties that may be calculated ("measured") on the entity itself.

The ________________________ of an entity refers to geometrical properties that arise from its embedding in a higher dimensional space.

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Tangent vectors to curves on a surface are thought of as intrinsic geometrical quantities.
For example, $\vec{X}_1$ and $\vec{X}_2$.

The metric $g_{ij} = \vec{X}_i \cdot \vec{X}_j$ is, therefore, intrinsic.

Normals to a surface are thought of as extrinsic geometrical quantities.
For example, $\vec{U}$.

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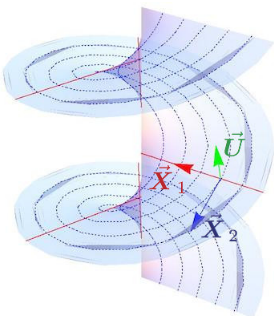
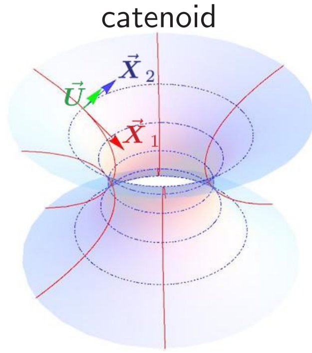
$\vec{X}_1$, $\vec{X}_2$, and $\vec{U}$ for the

helicoid



catenoid



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2. Raising and Lowering Indices

Suppose you have a quantity that looks like

$$Q_{\cdots}^{\cdots k}$$

where each dot is a possible index.

We "lower" the index k via

$$Q_{\cdots k}^{\cdots} \equiv Q_{\cdots}^{\cdots m} g_{mk}$$

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Indices are always "lowered" via the metric, g_{ij}, or "raised" via its inverse g^{ij}.

Although it's convenient to call both quantities "Q" each set of components may be different.

Example: $V^k = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $g_{ij} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

Then $V_j = g_{jk} V^k = (3, 8)$.

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OK, so you've lowered an index $Q_{\cdots k}^{\cdots} \equiv Q_{\cdots}^{\cdots m} g_{mk}$.

Then you think, "OMG, big mistake."
Let me raise it right back.

Math is Life: you can't turn back the clock.
(1) What do you do (mathematically)?

your $Q_{\cdots}^{\cdots k} =$

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ADDITIONAL NOTES

(2) Is your $Q^{\cdots k}$ = my $Q^{\cdots k}$?

your $Q^{\cdots k}_{\cdots} =$

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3. More about L_{ij}

The first fundamental form (metric) is intrinsic.

The second fundamental form

$$L_{ij} \equiv \vec{\boldsymbol{U}} \cdot X_{ij},$$

defined via the unit normal, $\vec{\boldsymbol{U}}$, would appear at first glance to be ____________.

We'll look at it in greater detail.

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Define

$$L^i_j = L_{jk} g^{ki}$$

where g^{ki} is the inverse of g_{jk} ($g_{jk} g^{ki} = \delta^j_i$).

(3) What's $L^i_j g_{im}$?

$L^i_j g_{im} =$

Also, define $\vec{\boldsymbol{U}}_i \equiv \dfrac{\partial \vec{\boldsymbol{U}}}{\partial u^i}$ (just as we defined $\vec{\boldsymbol{X}}_i$).

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(4) Fill in these blanks:

$\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{U}} =$ __, therefore, $\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{U}}_j =$ __.

Therefore, $\vec{\boldsymbol{U}}_j$ being orthogonal to the normal, is a linear combination of X_1 and X_2:

$$U_j = a^i_j \vec{\boldsymbol{X}}_i.$$

We'll determine the coefficients a^i_j.

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Now, $\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{X}}_k =$ __, therefore,

$\dfrac{\partial(\vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{X}}_k)}{\partial u^j} =$

Therefore,

$-L_{jk} =$

(5) How to solve for a^i_j? ____________

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$-L^m_j \equiv -$

Therefore

$$\boxed{\vec{\boldsymbol{U}}_j = -L^i_j \vec{\boldsymbol{X}}_i}$$ ("Weingarten's equations").

(6) Write out these equations in full components.

$\vec{\boldsymbol{U}}_1 =$

$\vec{\boldsymbol{U}}_2 =$

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ADDITIONAL NOTES

(7) What is $\vec{\boldsymbol{U}}_1 \times \vec{\boldsymbol{U}}_2$?

$\vec{\boldsymbol{U}}_1 \times \vec{\boldsymbol{U}}_2 =$

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(8) Why is $det(L_i^j) = K$?

1) $det(L_i^j) =$

2) If g $= det(g_{ij})$, then $det(g^{kj}) =$

3) So $det(L_i^j) =$

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4. More about Γ_{ij}^k

The quantities Γ_{ij}^k are called ______________________ _________________________________.

(9) ___________________________________?

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Now, Γ_{ij}^k were the coefficients of the tangential component of $\vec{\boldsymbol{X}}_{ij}$.

Since $\vec{\boldsymbol{X}}_{ij}$ is symmetric in $\{i, j\}$, so is Γ_{ij}^k:

$$\vec{\boldsymbol{X}}_{ij} = \Gamma_{ij}^k \vec{\boldsymbol{X}}_k + L_{ij}\vec{\boldsymbol{U}}$$

(10) “Dot” both sides with $\vec{\boldsymbol{X}}_l$.

$\vec{\boldsymbol{X}}_{ij} \cdot \vec{\boldsymbol{X}}_l =$

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(11) How does that help?

(12) What's $\dfrac{\partial g_{ji}}{\partial u^k} =$

(13) What's $\dfrac{\partial g_{kj}}{\partial u^i} =$

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(14) What's this?

(15) So

$\Gamma_{ijk} =$

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ADDITIONAL NOTES

(16) What's

$$\Gamma^m_{ij} = g^{mk}\Gamma_{ijk} =$$

(17) Does this make Γ^i_{jk} intrinsic or extrinsic?

____

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5. The Curvature "Tensor"

Start with our two boxed equations:

$$\boxed{\vec{X}_{ij} = \Gamma^k_{ij}\vec{X}_k + L_{ij}\vec{U}} \quad \text{(Gauss)}$$

$$\boxed{\vec{U}_j = -L^i_j\vec{X}_i} \quad \text{(Weingarten)}$$

where L_{ij} is the second fundamental form and $L^i_j = g^{jk}L_{ki}$. We now explore Γ^k_{ij}.

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Define $\vec{X}_{ijk} \equiv \dfrac{\partial X_{ij}}{\partial u^k} = \dfrac{\partial^3 \vec{X}}{\partial u^k \partial u^j \partial u^i}$

(18) Calculate X_{ijk} from Gauss's box.

$$\vec{X}_{ijk} =$$

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(19) Use Gauss's box to rewrite

$$\Gamma^n_{ij}\vec{X}_{nk} =$$

(20) Use Weingarten's box to rewrite

$$L_{ij}\vec{U}_k =$$

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Therefore,

$$\begin{aligned}\vec{X}_{ijk} = {} & \frac{\partial \Gamma^m_{ij}}{\partial u^k}\vec{X}_m \\ & + \Gamma^n_{ij}\Gamma^m_{nk}\vec{X}_m + \Gamma^m_{ij}L_{mk}\vec{U} \\ & + \frac{\partial L_{ij}}{\partial u^k}\vec{U} \\ & - L_{ij}L^m_k\vec{X}_m\end{aligned}$$

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(21) Rewrite this, grouping the $\vec{X}_m$ and $\vec{U}$ terms.

$$\vec{X}_{ijk} = \Big(\qquad\qquad\qquad\qquad \Big)\vec{X}_m + \Big(\qquad\qquad\qquad \Big)\vec{U}$$

(22) Apart from the exhilaration of algebra, what has this achieved? ________________________

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ADDITIONAL NOTES

(23) Switching indices is fun; rewrite with $j \leftrightarrow k$.

$$\vec{X}_{ikj} = \Big(\qquad\qquad\qquad\qquad \Big) \vec{X}_m + \Big(\qquad\qquad\qquad \Big) \vec{U}$$

(24) Remembering how $\vec{X}_{ijk}$ is defined, how is it related to $\vec{X}_{ikj}$? ________________

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(25) Set the "Tan" components equal to each other then regroup with all Γ terms on the left and all L on the right.

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The quantity

$$R^m_{ikj} \equiv \frac{\partial \Gamma^m_{ij}}{\partial u^k} - \frac{\partial \Gamma^m_{ik}}{\partial u^j} + \Gamma^n_{ij}\Gamma^m_{nk} - \Gamma^n_{ik}\Gamma^m_{nj}$$

is called the Reimann-Christoffel curvature tensor.

(26) Is it intrinsic or extrinsic?

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6. The *Theorema Egregium*

So, we have

$$R^m_{ikj} = L_{ij}L^m_k - L_{ik}L^m_j$$

Defining

$$R_{likj} \equiv g_{ml}R^m_{ikj}$$

we have

$$R_{likj} = L_{ij}L_{kl} - L_{ik}L_{jl}$$

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(27) What are R_{ij11}, R_{11ij}, R_{ij22}, and R_{22ij}?

______

(28) What's R_{1212}?
$(= R_{2121} = -R_{2112} = -R_{1221})$

(29) Therefore, $K =$ ______

That's the remarkable theorem.

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ADDITIONAL NOTES

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