Arvind Borde / MTH 675, Unit 12: The Second Fundamental Form

1 Acceleration and its Decomposition

Let $\boldsymbol{\alpha}(u^1(s), u^2(s))$ be a curve on surface $\mathcal{M}$, where s is the arc length. The tangent

$$\vec{\boldsymbol{T}} = \boldsymbol{\alpha}' = \frac{\partial \vec{\boldsymbol{X}}}{\partial u^1}\frac{du^1}{ds} + \frac{\partial \vec{\boldsymbol{X}}}{\partial u^2}\frac{du^2}{ds} \equiv u^{i\prime}\vec{\boldsymbol{X}}_i$$

may be viewed as the velocity of the curve.

We'd defined the curvature of $\boldsymbol{\alpha}$ as $\|\vec{\boldsymbol{T}}'\| = \|\boldsymbol{\alpha}''\|$.

1 We can think of $\boldsymbol{\alpha}''$ as the acceleration of $\boldsymbol{\alpha}$.

Now,
$$\boldsymbol{\alpha}'' = u^{i\prime\prime}\vec{\boldsymbol{X}}_i + u^{i\prime}\frac{d\vec{\boldsymbol{X}}_i}{ds},$$

where
$$\frac{d\vec{\boldsymbol{X}}_i}{ds} = \frac{\partial \vec{\boldsymbol{X}}_i}{\partial u^1}\frac{du^1}{ds} + \frac{\partial \vec{\boldsymbol{X}}_i}{\partial u^2}\frac{du^2}{ds}$$
$$= \frac{\partial^2 \vec{\boldsymbol{X}}}{\partial u^1 \partial u^i}u^{1\prime} + \frac{\partial^2 \vec{\boldsymbol{X}}}{\partial u^2 \partial u^i}u^{2\prime}$$
$$= \frac{\partial^2 \vec{\boldsymbol{X}}}{\partial u^j \partial u^i}u^{j\prime}$$

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If we define

$$\vec{\boldsymbol{X}}_{ji} = \frac{\partial^2 \vec{\boldsymbol{X}}}{\partial u^j \partial u^i} \quad \left(= \frac{\partial^2 \vec{\boldsymbol{X}}}{\partial u^i \partial u^j} = \vec{\boldsymbol{X}}_{ij}\right).$$

we get

$$\boldsymbol{\alpha}'' = u^{i\prime\prime}\vec{\boldsymbol{X}}_i + u^{i\prime}u^{j\prime}\vec{\boldsymbol{X}}_{ij}$$

Note: repeated indices that are summed over are

3 "dummy indices:" $A_iB^i = A_jB^j = A_kB^k$, etc.

We may decompose the vector $\vec{\boldsymbol{X}}_{ij}$ into a component tangent to surface $\mathcal{M}$ (hence a linear combination of $\vec{\boldsymbol{X}}_1$ and $\vec{\boldsymbol{X}}_2$), and a component normal to it (hence a multiple of the unit normal $\vec{\boldsymbol{U}}$):

$$\boxed{\vec{\boldsymbol{X}}_{ij} = \Gamma^k_{ij}\vec{\boldsymbol{X}}_k + L_{ij}\vec{\boldsymbol{U}}}$$

(No different from constructing a linear combination of the type $a^1\vec{\boldsymbol{X}}_1 + a^2\vec{\boldsymbol{X}}_2 + b\vec{\boldsymbol{U}}$, except that

4 we need such a linear combo for each of the $\vec{\boldsymbol{X}}_{ij}$.)

Plugging this into the equation for $\boldsymbol{\alpha}''$ we get

$$\boldsymbol{\alpha}'' = u^{i\prime\prime}\vec{\boldsymbol{X}}_i + u^{i\prime}u^{j\prime}\vec{\boldsymbol{X}}_{ji}$$
$$=$$
$$=$$

Define $\boldsymbol{\alpha}''_{\text{tan}} = \left(u^{k\prime\prime} + \Gamma^k_{ij}u^{i\prime}u^{j\prime}\right)\vec{\boldsymbol{X}}_k$, and

$$\boldsymbol{\alpha}''_{\text{nor}} = (L_{ij}u^{i\prime}u^{j\prime})\vec{\boldsymbol{U}}.$$

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2. The Second Fundamental Form

We'll discuss $\boldsymbol{\alpha}''_{\text{nor}}$ first.

Because it is a multiple of the unit normal, $\vec{\boldsymbol{U}}$, it captures properties of the surface $\mathcal{M}$ related to *how it is embedded in* $\mathbb{R}3$.

$L_{ij}u^{i\prime}u^{j\prime}$ is called the ______________________ of the surface $\mathcal{M}$.

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ADDITIONAL NOTES

(1) Taking the dot product of $\vec{\boldsymbol{U}}$ with the boxed equation, what do you get?

So, if you know $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{X}}_{ij}$ you can get L_{ij}. And, if you know the vectors $\vec{\boldsymbol{X}}_i$, you can get $\vec{\boldsymbol{X}}_{ij}$ by taking the second derivative.

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Example: Our sphere of radius R.

$$\vec{\boldsymbol{X}}_1 = R(-\sin u^1 \cos u^2, \cos u^1 \cos u^2, 0)$$
$$\vec{\boldsymbol{X}}_2 = R(-\cos u^1 \sin u^2, -\sin u^1 \sin u^2, \cos u^2)$$
$$\vec{\boldsymbol{U}} = (\cos u^1 \cos u^2,\ \sin u^1 \cos u^2,\ \sin u^2)$$
$$\vec{\boldsymbol{X}}_{11} =$$
$$\vec{\boldsymbol{X}}_{22} =$$
$$\vec{\boldsymbol{X}}_{12} =$$

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Therefore, for a sphere of radius R,

$$L_{11} = \vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{X}}_{11} =$$
$$L_{22} = \vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{X}}_{22} =$$
$$L_{12} = \vec{\boldsymbol{U}} \cdot \vec{\boldsymbol{X}}_{12} =$$

We'll see what L_{ij} is good for in a bit.

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3. Normal Curvatures

Let $\vec{\boldsymbol{V}} = V^i \vec{\boldsymbol{X}}_i$ be a unit tangent vector to $\mathcal{M}$ at a point **P**. We define the ________________ of $\mathcal{M}$ at **P** in the $\vec{\boldsymbol{V}}$ direction, $k_n(\vec{\boldsymbol{V}})$, as

$$k_n(\vec{\boldsymbol{V}}) = L_{ij} V^i V^j.$$

(2) Something should smell (faintly) fishy to you in this state of $k_n$mark.

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$\alpha_{\mathbf{V}}$ will have some curvature at the point **P**, $k_{\mathbf{P}}$.

There will be a normal to the surface $\mathcal{M}$ at **P**, $\vec{U}$.

We define the normal curvature of $\mathcal{M}$ at **P** in the $\vec{V}$ direction, $k_{\mathbf{U}}(\vec{\boldsymbol{V}})$, as

$$k_{\mathbf{U}}(\vec{\boldsymbol{V}}) \equiv \begin{cases} +k_{\mathbf{P}}, & \text{if } \vec{\boldsymbol{U}} = \vec{\boldsymbol{N}} \\ -k_{\mathbf{P}}, & \text{if } \vec{\boldsymbol{U}} = -\vec{\boldsymbol{N}} \end{cases}$$

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Let $\boldsymbol{\alpha}(s)$ be a curve on $\mathcal{M}$ such that $\mathbf{P} = \boldsymbol{\alpha}(s_0)$ and $\boldsymbol{\alpha}'(s_0) = \vec{\boldsymbol{V}} = V^i \vec{\boldsymbol{X}}_i$.

Since $\boldsymbol{\alpha}'(s_0) = u^{i\prime}(s_0)\vec{\boldsymbol{X}}_i$, $V^i = u^{i\prime}(s_0)$, and

$$k_n(\vec{\boldsymbol{V}}) = L_{ij} u^{i\prime} u^{j\prime} = \boldsymbol{\alpha}'' \cdot \vec{\boldsymbol{U}}.$$

Remember we'd defined a "normal section" of $\mathcal{M}$, as the curve at the intersection of $\mathcal{M}$ with a plane.

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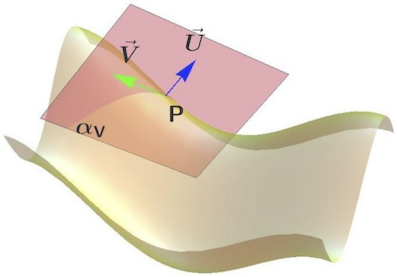
ADDITIONAL NOTES

The plane was one spanned by an arbitrary choice of tangent vector $\vec{\boldsymbol{V}}$ at point $\mathbf{P} \in \mathcal{M}$ and the unit normal, $\vec{\boldsymbol{U}}$ at that point:



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Every normal section of $\mathcal{M}$ at any point $\mathbf{P} \in \mathcal{M}$ (a slicing of $\mathcal{M}$ by a plane *containing the normal,* $\vec{\boldsymbol{U}}$*, at* $\mathbf{P}$) is a curve in $\mathcal{M}$.

(3) Is every curve in $\mathcal{M}$ a normal section?

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If $\boldsymbol{\alpha}$ happens to be a normal section, then we've seen that $\boldsymbol{\alpha}'' = \pm\|\boldsymbol{\alpha}''\|\vec{\boldsymbol{U}}$.

Thus, for such curves, our new "normal curvature of $\mathcal{M}$ at $\mathbf{P}$ in the $\vec{\boldsymbol{V}}$ direction," $k_n(\vec{\boldsymbol{V}}) = \boldsymbol{\alpha}'' \cdot \vec{\boldsymbol{U}}$, reduces to our old definition $\|\boldsymbol{\alpha}''\|$.

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The original definition, $k_n(\vec{\boldsymbol{V}}) = L_{ij}V^iV^j$, was given for unit vectors $\vec{\boldsymbol{V}}$.

If $\vec{\boldsymbol{V}}$ is non-unit, we can use $\vec{\boldsymbol{V}}/\|\vec{\boldsymbol{V}}\|$, which is unit, in the formula:

$$k_n(\vec{\boldsymbol{V}}) = \frac{L_{ij}V^iV^j}{\|\vec{\boldsymbol{V}}\|^2} = \frac{L_{ij}V^iV^j}{g_{kl}V^kV^l},$$

where we've used $\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{W}} = g_{kl}V^kW^l$ (U11S2).

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(4) With this extended definition, what is $k_n(x\vec{\boldsymbol{V}})$, where $x \in \mathbb{R}$ is any nonzero scalar, in terms of $k_n(\vec{\boldsymbol{V}})$?

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4. Principal Curvatures

As discussed earlier, at each point $\mathbf{P} \in \mathcal{M}$, $k_n(\vec{\boldsymbol{V}})$ will have maximum and minimum values

$$k_1 = k_n(\vec{\boldsymbol{V}}_1) \quad \text{and} \quad k_2 = k_n(\vec{\boldsymbol{V}}_2)$$

called the ____________________ at $\mathbf{P}$. The vectors $\vec{\boldsymbol{V}}_1$ and $\vec{\boldsymbol{V}}_2$ at which these values are attained are called the ____________________ at $\mathbf{P}$.

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ADDITIONAL NOTES

Claim: $\vec{\boldsymbol{V}}_1 \cdot \vec{\boldsymbol{V}}_2 = 0.$

Let's denote $k_n(\vec{\boldsymbol{V}})$ by $k(V^1, V^2)$.

$$k(V^1, V^2) = \frac{L_{ij}V^iV^j}{g_{kl}V^kV^l}$$

In a principal normal direction we must have

$$\frac{\partial k}{\partial V^1} = 0 = \frac{\partial k}{\partial V^2}.$$

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(5) For any symmetric matrix A_{ij} that does not depend on V^i, what is $\frac{\partial}{\partial V^k}(A_{ij}V^iV^j)$? ______

(6) With that, and the quotient rule,

$$\frac{\partial k}{\partial V^m} =$$

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(7) What does the definition of k tell us?

$$L_{ij}V^iV^j =$$

(8) Plug this into the expression for $\partial k/\partial V^m$.

$$\frac{\partial k}{\partial V^m} =$$

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(9) At each extremum k_1 and k_2, achieved in directions (V_1^1, V_1^2) and (V_2^1, V_2^2), we must have

$$= 0$$

$$= 0$$

In other words

$L_{mj}V_1^j =$ ________ and $L_{mj}V_2^j =$ ________.

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(10) Therefore,

$$L_{mj}V_1^jV_2^m =$$

$$L_{mj}V_2^jV_1^m =$$

From the symmetry of L_{ij} the left hand sides of the two equations are ____________.

Further, the expressions in parentheses on the right are both ______________.

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Thus $(k_1 - k_2)\vec{\boldsymbol{V}}_1 \cdot \vec{\boldsymbol{V}}_2 =$ __.

For $k_1 \neq k_2$ we get the required orthogonality of $\vec{\boldsymbol{V}}_1$ and $\vec{\boldsymbol{V}}_2$.

For $k_1 = k_2$ we must have $k_n(\vec{\boldsymbol{V}}) =$ constant, and any two orthogonal directions may be chosen as $\vec{\boldsymbol{V}}_1$ and $\vec{\boldsymbol{V}}_2$.

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ADDITIONAL NOTES

5. The Gauss Curvature

As we've seen, the Gauss curvature of $\mathcal{M}$ is defined as the product of the two principal curvatures:

$$K = k_1 k_2$$

How do we calculate it?

Q 9 says $det(L_{mj} - k_\beta g_{mj}) = 0$, $\beta = \{1, 2\}$.

(11) Why?

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(12) Calculate $det(L_{mj} - k_\beta g_{mj})$.

$$\begin{Vmatrix} L_{11} - k_\beta g_{11} & L_{12} - k_\beta g_{12} \\ L_{21} - k_\beta g_{21} & L_{22} - k_\beta g_{22} \end{Vmatrix}$$

$=$

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So,

$$k_\beta^2 - k_\beta \frac{(L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11})}{\mathsf{g}} + \frac{\mathsf{L}}{\mathsf{g}} = 0$$

In other words, k_1 and k_2 are the two solutions to a quadratic equation of the form $k^2 + bk + c = 0$, where $c = k_1 k_2$ (because lhs can be factored into $(k - k_1)(k - k_2)$).

(13) Therefore, $K = k_1 k_2 =$.

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Summary of how to get the Gauss curvature, K:

- Get the first fundamental form (metric), g_{ij}.
- Get the second fundamental form, L_{ij}.
- K is the ratio of the determinants of the second and first fundamental forms, L/g .

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6. Examples

A] Sphere of radius R:

$g_{ij} = \begin{pmatrix} R^2 \cos^2 u^2 & 0 \\ 0 & R^2 \end{pmatrix}$ So, g =

$L_{ij} = \begin{pmatrix} & 0 \\ 0 & \end{pmatrix}$ So, L =

Therefore,

$$K = \quad .$$

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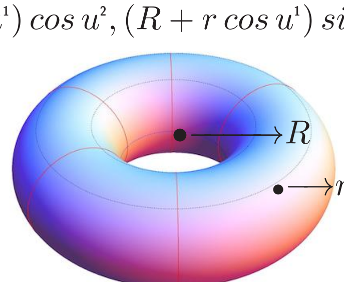
B] Torus:

$\vec{X}(u^1, u^2) =$

$((R + r\cos u^1)\cos u^2, (R + r\cos u^1)\sin u^2, r\sin u^1)$



$0 < u^1 < 2\pi$ (red, solid lines), $0 < u^2 < 2\pi$

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ADDITIONAL NOTES

Previously on the torus:

$$\vec{X}_1 = (-r\,sin\,u^1\,cos\,u^2, -r\,sin\,u^1\,sin\,u^2, r\,cos\,u^1)$$
$$\vec{X}_2 = (-(R + r\,cos\,u^1)\,sin\,u^2,\ (R + r\,cos\,u^1)\,cos\,u^2, 0)$$
$$\vec{U} = -(cos\,u^1\,cos\,u^2, cos\,u^1\,sin\,u^2, sin\,u^1)$$
$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\,cos\,u^1)^2 \end{pmatrix}$$

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$$\vec{X}_{11} =$$
$$\vec{X}_{22} =$$
$$\vec{X}_{12} =$$
$$L_{11} = \vec{U} \cdot \vec{X}_{11} =$$
$$L_{22} = \vec{U} \cdot \vec{X}_{22} =$$
$$L_{12} = \vec{U} \cdot \vec{X}_{12} =$$

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So,

g =

L =

$K =$

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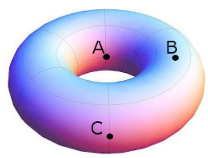
(14) What are the curvatures at these points?

A (“inside,” $u^1 = \pi$):

B (“on the top,” $u^1 = \pi/2$):

C (“outside, ” $u^1 = \pi$):



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C] Helicoid:

$$\vec{X}(u^1, u^2) = (u^1\,cos\,u^2, u^1\,sin\,u^2, bu^2)$$
$$\vec{X}_1 = (cos\,u^2, sin\,u^2, 0)$$
$$\vec{X}_2 = (-u^1\,sin\,u^2, u^1\,cos\,u^2, b)$$
$$\vec{U} = \frac{1}{\sqrt{b^2 + u^{1 2}}}(b\,sin\,u^2, -b\,cos\,u^2, u^1)$$
$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & b^2 + u^{1 2} \end{pmatrix}$$

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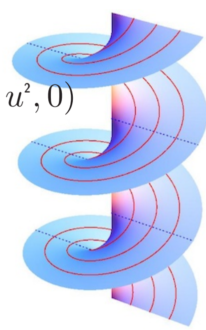
$$\vec{X}_{11} = (0, 0, 0)$$
$$\vec{X}_{22} = (-u^1\,cos\,u^2, -u^1\,sin\,u^2, 0)$$
$$\vec{X}_{12} = (-\,sin\,u^2, cos\,u^2, 0)$$
$$L_{11} = \vec{U} \cdot \vec{X}_{11} = 0$$
$$L_{22} = \vec{U} \cdot \vec{X}_{22} = 0$$
$$L_{12} = \vec{U} \cdot \vec{X}_{12} = \frac{-b}{\sqrt{b^2 + u^{1 2}}}$$
$$K = -b^2/(b^2 + u^{1 2})^2$$



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ADDITIONAL NOTES