Arvind Borde / MTH 675, Unit 11: What's the (F…F…F…) Metric?

1. The Einstein Summation Convention

The dot product of vectors

$$\vec{\boldsymbol{V}} = v^1\vec{\boldsymbol{X}}_1 + v^2\vec{\boldsymbol{X}}_2$$
$$\vec{\boldsymbol{W}} = w^1\vec{\boldsymbol{X}}_1 + w^2\vec{\boldsymbol{X}}_2$$

may be written as

$$\sum_{i,j} g_{ij}v^iw^j = v^1w^1g_{11} + (v^1w^2 + v^2w^1)g_{12} + v^2w^2g_{22}$$

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Einstein suggested that sums over repeated upper & lower indices be assumed, allowing us to write $g_{ij}v^iw^j$ with an assumed summation. We extend this idea to all formulas. From Unit 10, slide 13:

The tangent to $\boldsymbol{\alpha}(t)$ is

$$\frac{d\boldsymbol{\alpha}(t)}{dt} = \frac{\partial\vec{\boldsymbol{X}}}{\partial u_1}\frac{du_1}{dt} + \frac{\partial\vec{\boldsymbol{X}}}{\partial u_2}\frac{du_2}{dt}$$
$$= u_1'\vec{\boldsymbol{X}}_1 + u_2'\vec{\boldsymbol{X}}_2$$

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Label the parameters $\{u_1, u_2\}$ as $\{u^1, u^2\}$. (Hey, they're just labels, and can go anywhere.) We can write the previous result as

$$\boldsymbol{\alpha}'(t) = u^{1\prime}\vec{\boldsymbol{X}}_1 + u^{2\prime}\vec{\boldsymbol{X}}_2$$
$$\equiv u^{i\prime}\vec{\boldsymbol{X}}_i$$

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2. The Metric as a Matrix

The metric $g_{ij} \equiv \vec{\boldsymbol{X}}_i \cdot \vec{\boldsymbol{X}}_j$ may be viewed as a (symmetric) matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

with $g_{12} = g_{21}$. Then

$$\|\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2\|^2 = g_{11}g_{22} - g_{12}^2 = det\, g \equiv \mathsf{g} \neq 0.$$

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Then g_{ij} will have an inverse. In 2d it's simply

$$g^{ij} \equiv \frac{1}{\mathsf{g}}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$$

Define ("Kronecker delta symbol")

$$\delta_i^j \equiv \begin{cases} 1, & \text{when } i = j \\ 0, & \text{when } i \neq j \end{cases}$$

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(1) Express the delta symbol as a matrix.

Then

$$g_{ik}g^{kj} \equiv \sum_k g_{ik}g^{kj} = \delta_i^j$$

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ADDITIONAL NOTES

3. Sphere of Radius R

$$\vec{\boldsymbol{X}}(u^1, u^2) = R(\cos u^1 \cos u^2, \sin u^1 \cos u^2, \sin u^2)$$

with $-\pi < u^1 < \pi, \quad -\pi/2 < u^2 < \pi/2$.

$$\vec{\boldsymbol{X}}_1 = R(-\sin u^1 \cos u^2, \cos u^1 \cos u^2, 0)$$

$$\vec{\boldsymbol{X}}_2 = R(-\cos u^1 \sin u^2, -\sin u^1 \sin u^2, \cos u^2)$$

and $\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2 =$

$$R^2(\cos u^1 \cos^2 u^2,\ \sin u^1 \cos^2 u^2,\ \cos u^2 \sin u^2)$$

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$\vec{\boldsymbol{X}}_1$ is tangent to the "u^1 lines" ($u^2 =$ constant), and $\vec{\boldsymbol{X}}_2$ is tangent to the "u^2 lines" ($u^1 =$ constant).

$\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2$ is normal to the surface. Let

$$\vec{\boldsymbol{U}} = \frac{\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2}{\|\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2\|}$$

be the unit normal.

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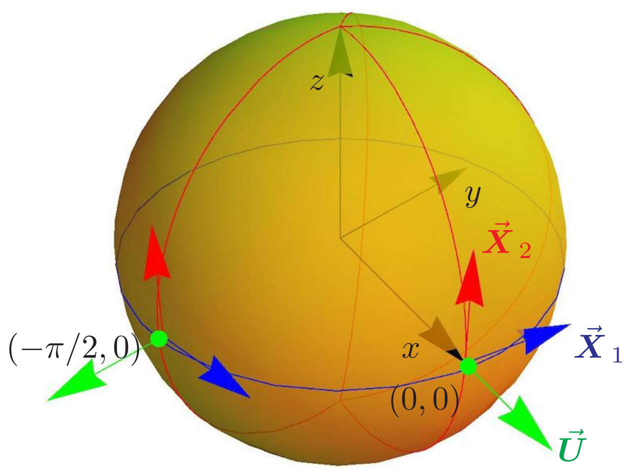

9

(2) What's the norm of the normal, $\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2$?

$\|\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2\|^2$

10

(3) What be $\vec{\boldsymbol{U}}$?

(4) Do 'dat $\vec{\boldsymbol{U}}$ be unit? ____________

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(5) What's the metric?

$g_{11} =$

$g_{22} =$

$g_{12} =$

Therefore

$ds^2 =$

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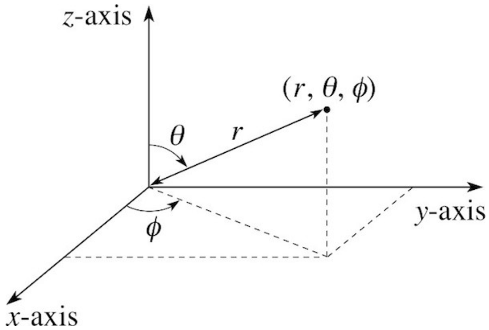
ADDITIONAL NOTES

How does $ds^2 = R^2 du^{2\,2} + R^2 \cos^2 u^2 du^{1\,2}$ compare with the usual result ($ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$)?



13

To summarize:

On a sphere of radius R, parametrized as given, the metric is

$$R^2 \begin{pmatrix} \cos^2 u^2 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$ds^2 = R^2 \cos^2 u^2 du^{1\,2} + R^2 du^{2\,2}$$

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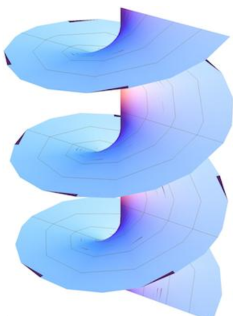
4. El Helicoide

$$\vec{\boldsymbol{X}}\,(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, bu^2)$$



$(b = 1, 0 < u^1 < 6, 0 < u^2 < 6\pi)$

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(6) Find $\vec{\boldsymbol{X}}_1$, $\vec{\boldsymbol{X}}_2$:

$\vec{\boldsymbol{X}}_1 =$

$\vec{\boldsymbol{X}}_2 =$

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(7) Find $\vec{\boldsymbol{U}}$:

$\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2 =$

$\|\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2\| =$

$\vec{\boldsymbol{U}} =$

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(8) What's the metric?

$g_{11} =$

$g_{22} =$

$g_{12} =$

Therefore

$ds^2 =$

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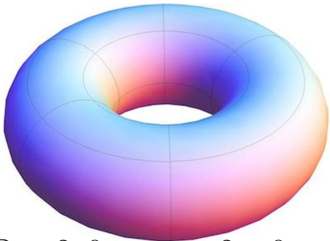
ADDITIONAL NOTES

5. Le Tore

$\vec{X}(u^1, u^2) =$
$\left((R + r\cos u^1)\cos u^2, (R + r\cos u^1)\sin u^2, r\sin u^1\right)$



19 $(r = 1, R = 2, 0 < u^1 < 2\pi, 0 < u^2 < 2\pi)$

(9) Find $\vec{X}_1$, $\vec{X}_2$:

$\vec{X}_1 =$

$\vec{X}_2 =$

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(10) Find $\vec{U}$:

$\vec{X}_1 \times \vec{X}_2 =$

$\|\vec{X}_1 \times \vec{X}_2\| =$

$\vec{U} =$

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(11) What's the metric?

$g_{11} =$

$g_{22} =$

$g_{12} =$

Therefore

$ds^2 =$

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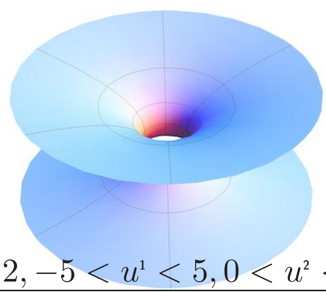
6. Das Katenoid

$\vec{X}(u^1, u^2) =$
$\left(a\cosh\left(\frac{u^1}{a}\right)\cos u^2, a\cosh\left(\frac{u^1}{a}\right)\sin u^2, u^1\right)$



23 $(a = 2, -5 < u^1 < 5, 0 < u^2 < 2\pi)$

(12) Find $\vec{X}_1$, $\vec{X}_2$:

$\vec{X}_1 =$

$\vec{X}_2 =$

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ADDITIONAL NOTES

(13) Find $\vec{\boldsymbol{U}}$:

$\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2 =$

$\|\vec{\boldsymbol{X}}_1 \times \vec{\boldsymbol{X}}_2\| =$

$\vec{\boldsymbol{U}} =$

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(14) What's the metric?

$g_{11} =$

$g_{22} =$

$g_{12} =$

Therefore

$ds^2 =$

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ADDITIONAL NOTES

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