Arvind Borde / MTH 675, Unit 10: The First Fundamental Form

1. Introduction

The manifold view allows us to discuss nd entities entirely on their own terms, without drawing upon how they might be embedded in $\mathbb{R}^{n+1}$.

For manifolds with $n \geqslant 3$ such a view is essential, especially, as in general relativity, if the manifold is meant to serve as a model for the Universe.

1 For some 2d surfaces such a view is essential, too.

On the one hand, surfaces such as spherical, planar, toroidal ones, etc., can all be viewed as embeddings in $\mathbb{R}^3$.

But there are surfaces, such as a Klein bottle, that cannot be correctly embedded in $\mathbb{R}^3$ (because they do not divide $\mathbb{R}^3$ into interior and exterior regions). To study the geometry of these, the manifold approach is necessary.

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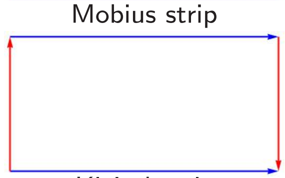
Klein bottle

Mobius strip



Klein bottle

3 https://s3files.core77.com/blog/images/2013/06/klein-bottle-01.jpg

Having said that, it's easier to visualize simpler 2d surfaces via their embeddings in $\mathbb{R}^3$, so that's how we'll begin.

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2. Parametrized Surfaces in $\mathbb{R}^3$

An elementary ______________ is a vector valued function of two real variables, $\{u_1, u_2\}$, defined on a set $(a_1, b_1) \times (a_2, b_2) \equiv D \subset \mathbb{R}^2$:

$$\equiv \big(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)\big)$$

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From the haze of Unit 3, slide 1:

> A ____curve____ is a vector-valued function of a real variable, t, defined on an interval $[a, b]$:
>
> $$\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$$
>
> We call t the ____parameter____ of the curve.

Following this, we'll call u_1, u_2 the ___________ of the surface.

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ADDITIONAL NOTES

We assume the coordinate functions have continuous partial derivatives up to third order, at least.

Further, we assume that the vectors

$$\vec{X}_1(u_1,u_2) \equiv \frac{\partial \vec{X}}{\partial u_1} = \left(\frac{\partial x}{\partial u_1}, \frac{\partial y}{\partial u_1}, \frac{\partial z}{\partial u_1}\right)$$
$$\vec{X}_2(u_1,u_2) \equiv \frac{\partial \vec{X}}{\partial u_2} = \left(\frac{\partial x}{\partial u_2}, \frac{\partial y}{\partial u_2}, \frac{\partial z}{\partial u_2}\right)$$

are linearly independent (i.e., not parallel).

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Example: $\vec{X}(u_1,u_2) =$
$(R\,cos\,u_1\,cos\,u_2, R\,sin\,u_1\,cos\,u_2, R\,sin\,u_2)$
with $-\pi < u_1 < \pi, \quad -\pi/2 < u_2 < \pi/2$.
(1) What's $\|\vec{X}\|$?

$\|\vec{X}\| =$

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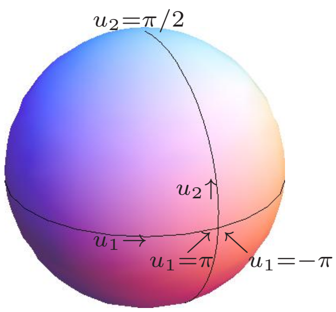
(2) Is the sphere fully covered by these coords? If not, which part isn't?



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(3) Calculate the following:

$\vec{X}_1 =$

$\vec{X}_2 =$

(4) Calculate $\vec{X}_1 \times \vec{X}_2$.

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(5) What are $\vec{X}_1$, $\vec{X}_2$, and $\vec{X}_1 \times \vec{X}_2$

- at $(0,0)$? $\vec{X}_1 =$
  $\vec{X}_2 =$
  $\vec{X}_1 \times \vec{X}_2 =$
- at $(-\pi/2,0)$? $\vec{X}_1 =$
  $\vec{X}_2 =$
  $\vec{X}_1 \times \vec{X}_2 =$

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3. Curves on Surfaces

Now that our hazy memory of curves is sharper, we define a curve on a surface, $\mathcal{M}$, in two steps:

1) A curve in D is a vector valued function of some parameter $t \in I \subset \mathbb{R}$: $\big(u_1(t), u_2(t)\big)$.

2) This curve in D will be mapped smoothly to a curve, $\boldsymbol{\alpha}(t)$, in surface $\mathcal{M}$, by the mapping $\vec{X}$:

$$\boldsymbol{\alpha}(t) = \vec{X}\big(u_1(t), u_2(t)\big)$$

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ADDITIONAL NOTES

The tangent to $\boldsymbol{\alpha}(t)$ is

$$\frac{d\boldsymbol{\alpha}(t)}{dt} =$$

where a prime is a derivative with respect to t.

Note: To calculate the tangent at a point t_0, $u_1{}'$ and $u_2{}'$ are calculated at t_0, and $\vec{\boldsymbol{X}}_1$ and $\vec{\boldsymbol{X}}_2$ at $\big(u_1(t_0), u_2(t_0)\big)$.

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We call $\vec{\boldsymbol{V}}_{\mathbf{P}}$ a tangent vector to $\mathcal{M}$ at **P** if there is a curve $\boldsymbol{\alpha}$ on $\mathcal{M}$ through **P** such that $\vec{\boldsymbol{V}} = \boldsymbol{\alpha}'$.

The set of all tangent vectors at **P** is called the tangent plane of $\mathcal{M}$ at **P**, and is denoted by $T_{\mathbf{P}}\mathcal{M}$.

Any tangent vector at a point $\vec{\boldsymbol{X}}(u_{10}, u_{20})$ is a linear combination of $\vec{\boldsymbol{X}}_1(u_{10}, u_{20})$ and $\vec{\boldsymbol{X}}_2(u_{10}, u_{20})$, as we've seen.

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Conversely, let

$$\vec{\boldsymbol{V}}(u_{10}, u_{20}) = a\vec{\boldsymbol{X}}_1(u_{10}, u_{20}) + b\vec{\boldsymbol{X}}_2(u_{10}, u_{20})$$

with $a, b \in \mathbb{R}$.

Consider the curve in D, $\big(u_1(t), u_2(t)\big)$, given by

$$u_1(t) = u_{10} + at, \qquad u_2(t) = u_{20} + bt,$$

and the curve in $\mathcal{M}$, $\boldsymbol{\alpha}(t) = \vec{\boldsymbol{X}}\big(u_1(t), u_2(t)\big)$.

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(6) What are $u_1{}'$ and $u_2{}'$? ____________________

(7) Therefore, what is $\boldsymbol{\alpha}'(u_{10}, u_{20})$?

____

In other words, at any point **P**, every tangent vector is a linear combination of $\vec{\boldsymbol{X}}_1$ and $\vec{\boldsymbol{X}}_2$, and every linear combination of $\vec{\boldsymbol{X}}_1$ and $\vec{\boldsymbol{X}}_2$ is a tangent vector.

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$\vec{\boldsymbol{X}}_1$ and $\vec{\boldsymbol{X}}_2$ are themselves tangent to curves. (8) Which?

____

____

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4. The First Fundamental Form

Returning to our youth again (Unit3, slide 22):

All of which motivates the definition of arc length, s, of a curve, $\boldsymbol{\alpha}(t)$, defined on $[a, b]$, from a to t:

$$s(t) = \int_a^t \|\boldsymbol{\alpha}'(t)\| dt$$

and, as we've also seen,

$$\frac{ds}{dt} =$$

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ADDITIONAL NOTES

(**9**) Obtain $\left(\dfrac{ds}{dt}\right)^2$ in terms of $\vec{\boldsymbol{X}}_1$, $\vec{\boldsymbol{X}}_2$, $u_1{}'$, $u_2{}'$.

$$\left(\frac{ds}{dt}\right)^2 =$$

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Define

$$g_{ij} \equiv \vec{\boldsymbol{X}}_i \cdot \vec{\boldsymbol{X}}_j$$

In terms of these, the previous formula becomes

$$\left(\frac{ds}{dt}\right)^2 = g_{11}\left(\frac{du_1}{dt}\right)^2 + 2g_{12}\left(\frac{du_1}{dt}\right)\left(\frac{du_2}{dt}\right) + g_{22}\left(\frac{du_2}{dt}\right)^2$$

often abbreviated ("differential" notation) as

$$ds^2 = g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2$$

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The expression on the right of either of the last two equations is called the ____________________ or the ________________ of the surface.

Many of the geometrical properties of the surface are encapsulated in it.

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The first fundamental form was introduced by Gauss in 1925–1927.

He'd used E for g_{11}, F for g_{12} and G for g_{22}:

it is clear that
$$\sqrt{(E\,dp^2 + 2F\,dp\,.\,dq + G\,dq^2)}$$
is the general expression for the linear element on the curved surface.

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DISQUISITIONES GENERALES

CIRCA

SUPERFICIES CURVAS

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE OBLATAE D. 8. OCTOB. 1827

COMMENTATIONES SOCIETATIS REGIAE SCIENTIARUM
GOTTINGENSIS RECENTIORES. VOL. VI. GOTTINGAE MDCCCXXVIII

GOTTINGAE
TYPIS DIETERICHIANIS
MDCCCXXVIII

Gauss: "General Investigations of Curved Surfaces," (1927)

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5. Geometry from the Metric

A] Arclengths

$$s(t) = \int_a^t \|\boldsymbol{\alpha}'(t)\| = \int_a^t \sqrt{\left(\frac{ds}{dt}\right)^2}\,dt$$

Or, symbolically,

$$s = \int_\alpha \sqrt{g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2}$$

Knowing g_{ij} allows you to find arc lengths.

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ADDITIONAL NOTES

B] Angles

From Unit 1, slide 15:

Two vectors $\vec{V}$ and $\vec{W}$ define a plane. In that plane, the angle, θ, between the two is

$$\cos\theta = \frac{\vec{V}}{\|\vec{V}\|} \cdot \frac{\vec{W}}{\|\vec{W}\|}$$

To see this, we use the law of cosines.

The norm, too, can be found from the dot product.

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Let $\vec{V} = v^1\vec{X}_1 + v^2\vec{X}_2$ & $\vec{W} = w^1\vec{X}_1 + w^2\vec{X}_2$, where a^i represents a component, not a power.

(10) Find $\vec{V} \cdot \vec{W}$ in terms of g_{ij}.

Knowing g_{ij} allows you to find dot products, hence norms and angles.

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C] Areas

Areas are related to cross products: the area of the parallelogram spanned by two independent vectors, $\vec{V}$ and $\vec{W}$, in the plane defined by them, is $\|\vec{V} \times \vec{W}\|$. From Unit 5, slide 3:

can show for any two vectors, $\vec{V}$ and $\vec{W}$ that

$$\|\vec{V} \times \vec{W}\|^2 = (\vec{V} \cdot \vec{V})(\vec{W} \cdot \vec{W}) - (\vec{V} \cdot \vec{W})^2$$

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(11) Calculate $\|\vec{X}_1 \times \vec{X}_2\|^2$.

Therefore, as long as $\vec{X}_1$, $\vec{X}_2$ are not parallel, $det\, g \neq 0$, and areas are (in principle) calculable from g_{ij}.

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ADDITIONAL NOTES

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